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Degenerate Two-Phase Incompressible Flow

I. Existence, Uniqueness and Regularity of a Weak Solution

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This is the first paper of a series in which we analyze mathematical properties and develop numerical methods for a degenerate elliptic-parabolic partial differential system which describes the flow of two incompressible, immiscible fluids in porous media. In this paper we first show that this system possesses a weak solution under physically reasonable hypotheses on the data. Then we prove that this weak solution is unique. Finally, we establish regularity on the weak solution which is needed in the uniqueness proof. In particular, the Hölder continuity of the saturation in space and time and the Lipschitz continuity of the pressure in space are obtained. © 2001 Academic Press

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1. INTRODUCTION

The differential equations describing the flow of two incompressible, immiscible fluids in porous media have been studied in the past few decades. Existence of weak solutions to these equations has been shown under various assumptions on physical data [2, 5, 6, 10, 21, 22]. Some of the papers assumed nondegeneracy of the phase mobilities (see, e.g., [21, 22]), while others required nonsingularity of the capillary pressure [2, 5, 10], for example. However, uniqueness of the weak solutions has been open, and their regularity has not been established. The degeneracy and strong coupling of the two-phase flow differential equations makes it hard to obtain the uniqueness and regularity of the solutions.

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In this paper we analyze a degenerate elliptic-parabolic partial differential system for the flow of two incompressible, immiscible fluids in porous media. Special attention is paid to the uniqueness and regularity of a weak solution to this system; for completeness, existence of the weak solution is also studied. Following [4, 10], we write this system in terms of a fractional flow formulation; i.e., a global pressure is used. However, rather than a saturation, a complementary pressure is introduced here. In this form, the system formally appears to be non-degenerate. More importantly, with it we are not only able to prove existence of a weak solution to this system with various boundary conditions under physically reasonable hypotheses on the data (the phase mobilities can be degenerate and the capillary pressure singular, for example), but also we can show uniqueness of the weak solution and obtain its certain regularity.

The existence analysis makes no use of the corresponding regularized problem; a weak solution is obtained as a limit of solutions to discrete time problems. It is similar to those used in [2, 3, 6] for treating quasilinear elliptic-parabolic differential equations. Especially, the approach we use follows that in [6], where a complementary pressure was also used. But, it is different from the one exploited here. Emphasis in [6] is put on the study of so-called dual-porosity models of two-phase flow in fractured porous media. The complementary pressure introduced here not only allows for a simpler proof of the existence, but also leads to the proof of the uniqueness.

Uniqueness results available so far for two-phase incompressible flow are those in [5, 22], where uniqueness of classical solutions to nondegenerate systems was shown. The uniqueness analysis here uses a constructive method, which only requires the global pressure to be Lipschitz continuous in space. To show the Lipschitz continuity of the pressure, we need the saturation to be Hölder continuous in space. The latter is established by adapting a technique utilized in [18] for handling degenerate parabolic equations. Here special care has to be taken on the degeneracy and strong coupling of the differential system under consideration. The Hölder continuity of the saturation in time is also obtained. We mention that a continuity result on the saturation was given in [2] under some conditions on the data.

The rest of the paper is organized as follows. In the next section we carefully state the assumptions on the data and briefly prove existence of a weak solution under these assumptions. Uniqueness and regularity of the weak solution is considered in sections three and four, respectively. We end with a remark that since the differential system for the single-phase, miscible displacement of one incompressible fluid by another in porous media resembles that for the two-phase incompressible flow studied here [13], the analysis presented in this paper extends to the miscible displacement problem. Numerical methods for solving the differential system under consideration and their analysis and implementation will be considered in the second and third papers of this series [14, 15].

2. EXISTENCE OF A WEAK SOLUTION

In this section we consider the flow of two incompressible, immiscible fluids in a porous medium $\Omega \subset \mathcal{R}^d$ $(d \leqslant 3)$. The flow equations will be reviewed in Subsection 2.1 and the differential system we study will be derived there. Then, in Subsection 2.2 we carefully state the assumptions on the physical data, define what is meant by a weak solution, and state the main result in this section. The proof of the main result is presented in Subsections 2.3 and 2.4.

2.1. *Flow Equations.* The mass balance equation for each of the fluid phases is given by ([9, 25])

$$\phi \frac{\partial(\rho_\alpha s_\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha u_\alpha) = \rho_\alpha q_\alpha, \qquad \alpha = w, n, \tag{2.1}$$

where $\alpha = w$ denotes the wetting phase (e.g. water), $\alpha = n$ indicates the nonwetting phase (e.g., oil or air), ϕ is the porosity of the porous medium, and ρ_α, s_α, u_α, and q_α are, respectively, the density, (reduced) saturation, volumetric velocity, and external volumetric flow rate of the α-phase. The volumetric velocity u_α is given by the Darcy law

$$u_\alpha = -\frac{\kappa \kappa_{r\alpha}}{\mu_\alpha} (\nabla p_\alpha - \rho_\alpha g), \qquad \alpha = w, n, \tag{2.2}$$

where κ is the absolute permeability of the porous medium, p_α, μ_α, and $\kappa_{r\alpha}$ are the pressure, viscosity, and relative permeability of the α-phase, respectively, and g denotes the gravitational, downward-pointing, constant vector. In addition to (2.1) and (2.2), the customary property for the saturations is

$$s_w + s_n = 1, \tag{2.3}$$

and the two pressures are related by the capillary pressure function

$$p_c = p_n - p_w. \tag{2.4}$$

In order to separate the pressure and saturation equations, we introduce the phase mobility functions

$$\lambda_\alpha(x, s_\alpha) = \kappa_{r\alpha}/\mu_\alpha, \qquad \alpha = w, n,$$

and the total mobility

$$\lambda(x, s) = \lambda_w + \lambda_n,$$

where $s = s_w$. The fractional flow functions are defined by

$$f_\alpha(x, s) = \lambda_\alpha/\lambda, \qquad \alpha = w, n.$$

Following [4, 10], we define the global pressure

$$p = p_n - \int_0^s \left(f_w \frac{\partial p_c}{\partial s} \right)(x, \xi)\, \mathrm{d}\xi. \tag{2.5}$$

Also, we shall use a complementary pressure (i.e., the Kirchhoff transformation)

$$\theta = -\int_0^s \left(f_w \lambda_n \frac{\partial p_c}{\partial s} \right)(x, \xi)\, \mathrm{d}\xi, \tag{2.6}$$

which is different from that used in [6] and needed in the proof of uniqueness. Note that while θ is called the complementary pressure [6], it does not have the unit of the pressure, so it is a pseudo-pressure. Finally, we define the total velocity

$$u = u_w + u_n. \tag{2.7}$$

Now, under the assumption that the fluids are incompressible we apply (2.3) and (2.7) to (2.1) to see that

$$\nabla \cdot u = q(s) \equiv q_w + q_n, \tag{2.8}$$

and (2.4), (2.5), and (2.7) to (2.2) to obtain

$$u = -\kappa(\lambda(s)\, \nabla p + \gamma_1(s)), \tag{2.9}$$

where

$$\gamma_1 = -\lambda_w \nabla_x p_c + \lambda \int_0^s \nabla_x \left(f_w \frac{\partial p_c}{\partial s} \right)(x, \xi)\, \mathrm{d}\xi - (\lambda_w \rho_w + \lambda_n \rho_n)\, g.$$

Similarly, apply (2.4)–(2.6) to (2.1) and (2.2) with $\alpha = w$ to have

$$\phi \frac{\partial s}{\partial t} - \nabla \cdot \{\kappa(\nabla\theta + \lambda_w(s)\, \nabla p + \gamma_2(s))\} = q_w(s), \tag{2.10}$$

where

$$\gamma_2 = -\lambda_w \nabla_x p_c + \lambda_w \int_0^s \nabla_x \left(f_w \frac{\partial p_c}{\partial s} \right)(x, \xi)\, d\xi$$
$$+ \int_0^s \nabla_x \left(f_w \lambda_n \frac{\partial p_c}{\partial s} \right)(x, \xi)\, d\xi - \lambda_w \rho_w g.$$

It can be seen that the phase velocities are determined by

$$\begin{aligned} u_w &= -\kappa(\nabla\theta + \lambda_w(s)\, \nabla p + \gamma_2(s)), \\ u_n &= \kappa(\nabla\theta - \lambda_n(s)\, \nabla p + \gamma_3(s)), \end{aligned} \tag{2.11}$$

where

$$\gamma_3 = -\lambda_n \int_0^s \nabla_x \left(f_w \frac{\partial p_c}{\partial s} \right)(x, \xi)\, d\xi + \int_0^s \nabla_x \left(f_w \lambda_n \frac{\partial p_c}{\partial s} \right)(x, \xi)\, d\xi + \lambda_n \rho_n g.$$

Finally, s is related to θ through (2.6),

$$s = \mathscr{S}(\theta), \tag{2.12}$$

where $\mathscr{S}(x, \theta)$ is the inverse of (2.6) for $0 \leqslant \theta \leqslant \theta^*(x)$ with

$$\theta^*(x) = -\int_0^1 \left(f_w \lambda_n \frac{\partial p_c}{\partial s} \right)(x, \xi)\, d\xi.$$

The pressure equation is given by (2.8) and (2.9), while the saturation equation is described by (2.10). They determine the main unknowns p, s, and θ. The model is completed by specifying boundary and initial conditions.

With the following division of the boundary Γ of Ω,

$$\Gamma = \Gamma_1^p \cup \Gamma_2^p = \Gamma_1^\theta \cup \Gamma_2^\theta, \qquad \varnothing = \Gamma_1^p \cap \Gamma_2^p = \Gamma_1^\theta \cap \Gamma_2^\theta,$$

the boundary conditions are specified by

$$\begin{aligned} &u \cdot v - a_1(s)\, p = \varphi_1(s), && (x, t) \in \Gamma_1^p \times J, \\ &p = \varphi_2(x, t), && (x, t) \in \Gamma_2^p \times J, \\ &u_w \cdot v - a_2(s)\, \theta = \varphi_3(s), && (x, t) \in \Gamma_1^\theta \times J, \\ &\theta = \varphi_4(x, t), && (x, t) \in \Gamma_2^\theta \times J, \end{aligned} \tag{2.13}$$

where the a_i and φ_i are given functions, $J = (0, T]$ $(T > 0)$, and v is the outer unit normal to Γ. The boundary conditions in (2.13) come from those imposed for the phase quantities via the transformations (2.5) and (2.6) [11, 12, 16]. The initial condition is given by

$$\theta(x, 0) = \theta_0(x), \qquad x \in \Omega. \tag{2.14}$$

The differential system has a clear structure. Note that while λ_w and λ_n can be zero, λ is always positive in practice. That is, the pressure equation is elliptic for p, and the saturation equation is parabolic for θ. This model has been analyzed from the computational point of view in [7, 8, 11, 12, 16, 25], for example.

2.2. *Assumptions and the Main Result.* The usual Sobolev spaces $W^{l,\pi}(\Omega)$ with the norm $\|\cdot\|_{W^{l,\pi}(\Omega)}$ ([1]) will be used, where l is a non-negative integer and $0 \leqslant \pi \leqslant \infty$. When $\pi = 2$, we simply write $H^l(\Omega) = W^{l,2}(\Omega)$. When $l = 0$, we have $L^2(\Omega) = H^0(\Omega)$. Below $(\cdot, \cdot)_Q$ denotes the $L^2(Q)$ inner product (Q is omitted if $Q = \Omega$). Also, set $\Omega_T = \Omega \times J$. We now make the following assumptions:

(A1) $\Omega \subset \mathscr{R}^d$ is a multiply connected domain with Lipschitz boundary Γ, $\Gamma = \Gamma_1^p \cup \Gamma_2^p = \Gamma_1^\theta \cup \Gamma_2^\theta$, $\Gamma_1^p \cap \Gamma_2^p = \Gamma_1^\theta \cap \Gamma_2^\theta = \varnothing$, each Γ_i^p and Γ_i^θ is a $(d-1)$-dimensional domain, and $\Gamma_2^p \subset \Gamma_2^\theta$.

(A2) $\phi \in L^\infty(\Omega)$ satisfies that $\phi(x) \geqslant \phi_* > 0$, and $\kappa(x)$ is a bounded, symmetric, and uniformly positive definite matrix; i.e.,

$$0 < \kappa_* \leqslant |\xi|^{-2} \sum_{i,j=1}^{d} \kappa_{ij}(x)\, \xi_i \xi_j \leqslant \kappa^* < \infty, \qquad x \in \Omega, \qquad \xi \neq 0 \in \mathscr{R}^d.$$

(A3) $\lambda_\alpha(x, s)$ is measurable in $x \in \Omega$ and continuous in $s \in [0, 1]$, and satisfies that $\lambda_w(0) = 0$, $\lambda_w(s) > 0$ for $s > 0$, $\lambda_n(1) = 0$, $\lambda_n(s) > 0$ for $s < 1$, and $0 < \lambda_* \leqslant \lambda(x, s, c) \leqslant \lambda^* < \infty$, $x \in \Omega$, $s \in [0, 1]$.

(A4) $\mathscr{S}\colon \{(x, \theta) : x \in \Omega, 0 \leqslant \theta \leqslant \theta^*(x)\} \to [0, 1]$ is measurable in x and continuous and strictly monotone increasing in θ, and satisfies that $\mathscr{S}(x, 0) = 0$ and $\mathscr{S}(x, \theta^*(x)) = 1$, where $0 < \theta^* \in H^1(\Omega)$.

(A5) γ_1, γ_2, φ_1, φ_3, q, and q_w are measurable in $x \in \Omega$ and continuous in s, and the following norms are bounded,

$$|||\gamma_1|||_{L^\infty(J;\, L^2(\Omega))}, \qquad |||\gamma_2|||_{L^2(\Omega_T)}, \qquad |||\varphi_1|||_{L^\infty(J;\, H^{-1/2}(\Gamma_1^p))},$$

$$|||q|||_{L^\infty(J;\, H^{-1}(\Omega))}, \qquad |||q_w|||_{L^2(J;\, H^{-1}(\Omega))}, \qquad |||\varphi_3|||_{L^2(J;\, H^{-1/2}(\Gamma_1^\theta))},$$

where for $v = v(x, s)$, the norm $|||\cdot|||$ is given by

$$|||v||| = \| \sup_{s \in [0,\, 1]} |v(x, s)| \|,$$

for any given norm $\|\cdot\|$. Furthermore, assume that

$$q_w(0) \geqslant 0, \qquad q_n(1) = q(1) - q_w(1) \geqslant 0 \qquad \text{on} \quad \Omega_T,$$

$$\varphi_1(1) \leqslant 0, \qquad \varphi_3(0) \leqslant 0, \qquad \varphi_3(1) \geqslant 0 \qquad \text{on} \quad \Gamma_1^\theta.$$

(A6) $\partial_t \varphi_4 \in L^1(\Omega_T)$ and

$$\varphi_2 \in L^\infty(J;\, H^1(\Omega)), \qquad \varphi_4 \in L^2(J;\, H^1(\Omega)),$$

$$0 \leqslant \varphi_4(x, t) \leqslant \theta^*(x) \qquad \text{a.e.} \quad \text{on } \Omega_T.$$

(A7) In the case of $\Gamma_2^p = \emptyset$ and $a_1 \equiv 0$, q and φ_1 are independent of s, and

$$\int_{\Gamma_1^p} \varphi_1 \, d\sigma = \int_\Omega q \, dx.$$

(A8) There is a subset $\Gamma_{1,*}^p \subset \Gamma_1^p$ (with nonzero measure only if $\Gamma_2^p = \emptyset$ and $a_1 \not\equiv 0$) such that $a_1 \geqslant a_{1,*} > 0$ on $\Gamma_{1,*}^p \times J$.

(A9) $a_i \geqslant 0$, a_i is continuous in s, the norm $|||a_i|||_{L^\infty(\Omega_T)}$ is bounded, $i = 1, 2$, and $a_1(1) = 0$ on Γ_1^θ.

(A10) $\theta_0 \in L^2(\Omega)$ satisfies that $0 \leqslant \theta_0 \leqslant \theta^*(x)$ a.e. on Ω.

All the ten assumptions are physically reasonable [9, 25]. Define the spaces

$$V = \left\{ v \in H^1(\Omega) : v|_{\Gamma_2^p} = 0;\ \text{if } \Gamma_2^p = \emptyset \text{ and } a_1 \equiv 0, \text{ then } \int_\Omega v \, dx = 0 \right\},$$

$$W = \{ v \in H^1(\Omega) : v|_{\Gamma_2^\theta} = 0 \}.$$

Below V^* and W^* indicate the duals of V and W, respectively.

DEFINITION 2.1. A weak solution of the system in (2.8)–(2.14) is a pair of functions (p, θ) with $p \in L^\infty(J; V) + \varphi_2$, $\theta \in L^2(J; W) + \varphi_4$ such that

$$s = \mathscr{S}(\theta), \qquad \phi\, \partial_t s \in L^2(J; W^*), \qquad 0 \leqslant \theta(x, t) \leqslant \theta^*(x) \qquad \text{a.e.} \quad \text{on } \Omega_T,$$

$$(\kappa\{\lambda(s)\, \nabla p + \gamma_1(s)\}, \nabla v) + (a_1(s)\, p, v)_{\Gamma_1^p} = (q(s), v) - (\varphi_1(s), v)_{\Gamma_1^p}, \quad \forall v \in L^\infty(J; V),$$

$$\int_J \langle \phi\, \partial_t s, v\rangle\, dt + \int_J (\kappa\{\nabla\theta + \lambda_w(s)\, \nabla p + \gamma_2(s)\}, \nabla v)\, dt + \int_J (a_2(s)\, \theta, v)_{\Gamma_1^\theta}\, dt$$

$$= \int_J (q_w(s), v)\, dt - \int_J (\varphi_3(s), v)_{\Gamma_1^\theta}\, dt, \qquad \forall v \in L^2(J; W),$$

$$\int_J \langle \phi\, \partial_t s, v\rangle\, dt + \int_J (\phi(s - s_0), \partial_t v)\, dt = 0,$$

$$\forall v \in L^2(J; W) \cap W^{1,1}(J; L^1(\Omega)), \qquad v(x, T) = 0,$$

where $s_0 = \mathscr{S}(\theta_0)$.

We now state the main result obtained in this section.

THEOREM 2.1. *Under assumptions* (A1)–(A10), *the system in* (2.8)–(2.14) *has a weak solution in the sense of Definition* 2.1.

2.3. *Proof of the Main Result.* As mentioned before, for completeness we shall briefly prove Theorem 2.1. The techniques we use are similar to those in [2, 3, 6]. In particular, the subsequent argument is related to that in [6]. However, thanks to (2.6), the proof presented here is much simpler than that in [6]. For each positive integer M, divide J into $m = 2^M$ subintervals of equal length $h = T/m = 2^{-M}T$. Set $t_i = ih$ and $J_i = (t_{i-1}, t_i]$ for an integer i, $1 \leqslant i \leqslant m$. Denote the time difference operator by

$$\partial^\eta v(t) = \frac{v(t+\eta) - v(t)}{\eta},$$

for any function $v(t)$ and constant $\eta \in \mathscr{R}$. Also, for any Hilbert space $\mathscr{H}$, define

$$l_h(\mathscr{H}) = \{v \in L^\infty(J; \mathscr{H}) : v \text{ is constant in time on each subinterval } J_i \subset J\}.$$

For $v^h \in l_h(\mathscr{H})$, set $v^i \equiv (v^h)^i = v^h|_{J_i}$ for notational convenience, when there is no ambiguity (i.e., h is omitted). Finally, let

$$\varphi^h(x, t) = \frac{1}{h} \int_{J_i} \varphi(x, \tau)\, d\tau, \qquad t \in J_i.$$

The discrete time solution is a pair of functions $p^h \in l_h(V) + \varphi_2^h$, $\theta^h \in l_h(W) + \varphi_4^h$ satisfying

$$\begin{aligned}(\kappa\{\lambda(s^h)\,\nabla p^h + \gamma_1(s^h)\}, \nabla v) + (a_1(s^h)\,p^h, v)_{\Gamma_1^p} \\ = (q(s^h), v) - (\varphi_1(s^h), v)_{\Gamma_1^p}, \qquad \forall v \in l_h(V),\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}&\int_J (\phi\,\partial^{-h} s^h, v)\,\mathrm{d}t + \int_J (\kappa\{\nabla\theta^h + \lambda_w(s^h)\,\nabla p^h + \gamma_2(s^h)\}, \nabla v)\,\mathrm{d}t \\ &\quad + \int_J (a_2(s^h)\,\theta^h, v)_{\Gamma_1^\theta}\,\mathrm{d}t \\ &= \int_J (q_w(s^h), v)\,\mathrm{d}t - \int_J (\varphi_3(s^h), v)_{\Gamma_1^\theta}\,\mathrm{d}t, \qquad \forall v \in l_h(W),\end{aligned} \tag{2.16}$$

where $s^h = \mathscr{S}(\theta^h)$ with $s^h = s_0$ when $t < 0$.

Below C (with or without a subscript) indicates a generic constant independent of h, which will probably take on different values in different occurrences.

LEMMA 2.1. *For $h > 0$ small enough, the discrete scheme has a solution such that*

$$0 \leqslant \theta^h(x, t) \leqslant \theta^*(x) \qquad \text{a.e. on } \Omega_T. \tag{2.17}$$

This lemma is purely an elliptic result and can be shown via a standard Galerkin approximation argument (see, e.g., [3, 5, 6]). The next lemma will be used in the proof of Lemma 2.3 below. For its proof, see [6].

LEMMA 2.2. *Let $b: \mathscr{R} \to \mathscr{R}$ be an increasing function such that $b(0) = 0$ and $c_0, c_1, \ldots$ be a sequence of real numbers. Then for any integer $m > 0$,*

$$\sum_{k=1}^{m} \{b(c_k) - b(c_{k-1})\}\,c_k \geqslant B(c_m) - B(c_0) \geqslant -B(c_0),$$

where

$$B(c) = \int_0^c \{b(c) - b(\tau)\}\,\mathrm{d}\tau.$$

LEMMA 2.3. *The solution to the discrete scheme also satisfies*

$$\|p^h\|_{L^\infty(J;\,H^1(\Omega))} + \|\theta^h\|_{L^2(J;\,H^1(\Omega))} \leqslant C. \tag{2.18}$$

Proof. Take $v = p^h - \varphi_2^h \in l_h(V)$ in (2.15) to have

$$\|\nabla p^h\|^2_{L^2(\Omega)} + (a_1(c^h)\, p^h, p^h)_{\Gamma_1^p} \leqslant C\,\{|||\gamma_1|||^2_{L^2(\Omega)} + |||q|||^2_{H^{-1}(\Omega)} + |||\varphi_1|||^2_{H^{-1/2}(\Gamma_1^p)} + \|\varphi_2^h\|^2_{H^1(\Omega)}\} + \varepsilon\, \|p^h\|^2_{L^2(\Omega)}, \qquad t \in J, \tag{2.19}$$

for any $\varepsilon > 0$. Apply the Poincare inequality

$$\|p^h\|_{L^2(\Omega)} \leqslant C\{\|\nabla p^h\|_{L^2(\Omega)} + \|p_h\|_{L^2(\Gamma^p_{1,*})} + \|\varphi_2^h\|_{H^1(\Omega)}\}, \tag{2.20}$$

and the obvious inequality

$$\|\varphi_2^h\|_{L^\infty(J;\, H^1(\Omega))} \leqslant \|\varphi_2\|_{L^\infty(J;\, H^1(\Omega))},$$

to obtain the bound for p^h in (2.18).

Next, choose $v = \theta^h - \varphi_4^h \in l_h(W)$ in (2.16) and use (2.17) and the above bound on p^h to see that

$$\begin{aligned}\int_J (\phi\, \partial^{-h} s^h, \theta^h - \varphi_4^h)\, dt + C_1 \int_J \|\nabla \theta^h\|^2_{L^2(\Omega)}\, dt \\ \leqslant C(T, \Omega) \Big\{ \|\theta^*\|^2_{L^2(\Omega)} + \int_J (|||q_w|||^2_{H^{-1}(\Omega)} + |||\varphi_3|||^2_{H^{-1/2}(\Gamma_1^c)} + \|\varphi_4^h\|^2_{H^1(\Omega)} \\ + |||q|||^2_{H^{-1}(\Omega)} + |||\varphi_1|||^2_{H^{-1/2}(\Gamma_1^p)} + \|\varphi_2^h\|^2_{H^1(\Omega)} \\ + |||\gamma_1|||^2_{L^2(\Omega)} + |||\gamma_2|||^2_{L^2(\Omega)})\, dt \Big\}.\end{aligned} \tag{2.21}$$

It can be seen from Lemma 2.2 that

$$\begin{aligned}\int_J (\phi\, \partial^{-h} s^h, \theta^h)\, dt &= \sum_{i=1}^{m} (\phi\{\mathscr{S}(\theta^i) - \mathscr{S}(\theta^{i-1})\}, \theta^i) \\ &\geqslant -\left(\phi, \int_0^{\theta^0} (\mathscr{S}(\theta^0) - \mathscr{S}(\tau))\, d\tau\right) \\ &\geqslant -C\, \|\theta^*\|_{L^1(\Omega)}.\end{aligned} \tag{2.22}$$

Also, we find that

$$\begin{aligned}\int_J (\phi\, \partial^{-h} s^h, \varphi_4^h)\, dt &= (\phi s^m, \varphi_4^m) - (\phi s^0, \varphi_4^1) - \int_0^{T-h} (\phi s^h, \partial^h \varphi_4^h)\, dt \\ &\leqslant C \left\{ \|\varphi_4^h\|_{L^\infty(J;\, L^1(\Omega))} + \int_0^{T-h} \|\partial^h \varphi_4^h\|_{L^1(\Omega)}\, dt \right\}.\end{aligned} \tag{2.23}$$

Finally, it is obvious that

$$\|\varphi_4^h\|_{L^2(J;\,H^1(\Omega))} \leqslant C\,\|\varphi_4\|_{L^2(J;\,H^1(\Omega))},$$
$$\|\varphi_4^h\|_{L^\infty(J;\,L^1(\Omega))} \leqslant C\,\|\varphi_4\|_{W^{1,1}(J;\,L^1(\Omega))}, \tag{2.24}$$

and

$$\begin{aligned}\int_0^{T-h} \|\partial^h \varphi_4^h\|_{L^1(\Omega)}\,\mathrm{d}t &= \sum_{i=1}^{n-1} \|\varphi_4^{i+1} - \varphi_4^i\|_{L^1(\Omega)} \\ &= \sum_{i=1}^{n-1} \frac{1}{h} \left\| \int_{t_{i-1}}^{t_i} \int_t^{t+h} \partial_t \varphi_4(\cdot,\tau)\,\mathrm{d}\tau\,\mathrm{d}t \right\|_{L^1(\Omega)} \\ &\leqslant \int_J \|\partial_t \varphi_4\|_{L^1(\Omega)}\,\mathrm{d}t. \end{aligned} \tag{2.25}$$

Now, combine (2.21)–(2.25) to have the desired result for θ^h. ∎

COROLLARY 2.1. *For any $2 \leqslant r_1 < \infty$, a subsequence exists such that $p^h \rightharpoonup p$ weakly in $L^{r_1}(J; H^1(\Omega))$ and $\theta^h \rightharpoonup \theta$ weakly in $L^2(J; H^1(\Omega))$ as $h \to 0^+$. Furthermore, $p \in L^\infty(J; V) + \varphi_2$, $\theta \in L^2(J; W) + \varphi_4$, and*

$$0 \leqslant \theta(x,t) \leqslant \theta^*(x) \qquad \text{a.e. on } \Omega_T. \tag{2.26}$$

Proof. It follows from Lemma 2.3 that $\theta^h - \varphi_4^h$ converges weakly in $L^2(J; W)$. Since $\varphi_4^h \rightharpoonup \varphi_4$ weakly in $L^2(J; H^1(\Omega))$, $\theta^h \rightharpoonup \theta$ weakly in $L^2(J; H^1(\Omega))$ with $\theta \in L^2(J; W) + \varphi_4$. The same argument shows that $p^h \rightharpoonup p$ weakly in $L^{r_1}(J; H^1(\Omega))$ with $p \in L^{r_1}(J; V) + \varphi_2$ for $2 \leqslant r_1 < \infty$. Since $\|p\|_{L^{r_1}(J;\,H^1(\Omega))} \leqslant C$ with C independent of r_1, in fact $p \in L^\infty(J; V) + \varphi_2$. Finally, (2.26) follows from (2.17). ∎

LEMMA 2.4. *There is a subsequence such that $\theta^h \to \theta$ strongly in $L^2(\Omega_T)$ as $h \to 0^+$.*

This lemma will be shown in the next subsection.

COROLLARY 2.2. *There is a subsequence such that $\theta^h \to \theta$ strongly in $L^2(J; H^{1-\pi}(\Omega))$ and $L^2(J; H^{1/2-\pi}(\partial\Omega))$ for any $0 < \pi < 1/2$, and $s^h \to s$ pointwisely a.e. on Ω_T, where $s = \mathscr{S}(\theta)$.*

Proof. Apply the interpolation inequality

$$\|v\|_{H^\sigma(\Omega)} \leqslant \delta\,\|v\|_{H^1(\Omega)} + C_\delta\,\|v\|_{L^2(\Omega)}, \tag{2.27}$$

for any $0<\sigma<1$ and $\delta>0$, the boundedness of the trace operator, and Lemma 2.4 to prove the first part of desired statement. The second part follows from the continuity of $\mathscr{S}(\theta)$ in θ. ∎

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. From Corollaries 2.1 and 2.2, the pressure equation in the weak sense of Definition 2.1 can be easily seen to hold since $\bigcup_{M=1}^{\infty} l_h(V)$ is dense in $L^\infty(J; V)$. Also, it follows from (2.16) that

$$\lim_{h\to 0^+} \int_J (\phi\,\partial^{-h} s^h, v)\,dt + \int_J (\kappa\{\nabla\theta + \lambda_w(s)\,\nabla p + \gamma_2(s)\}, \nabla v)\,dt$$
$$+ \int_J (a_2(s)\,\theta, v)_{\Gamma_1^\theta}\,dt$$
$$= \int_J (q_w(s), v)\,dt - \int_J (\varphi_3(s), v)_{\Gamma_1^\theta}\,dt, \qquad \forall v \in \bigcup_{M=1}^{\infty} l_h(W). \quad (2.28)$$

For any $v\in L^2(J; W)$, $v^h \in l_h(W)$, where $v^h(x,t) = h^{-1}\int_{J_k} v(x,\tau)\,d\tau$. Then, by (2.16), we observe that

$$\int_J (\phi\,\partial^{-h} s^h, v)\,dt = \int_J (\phi\,\partial^{-h} s^h, v^h)\,dt \leqslant C\,\|v\|_{L^2(J;\,H^1(\Omega))}.$$

Consequently, for a subsequence $\phi\,\partial^{-h} s^h$ converges weakly in $L^2(J; W^*)$. If $v\in C_0^\infty(\Omega_T)$, with $h>0$ small enough we see that

$$\int_J (\phi\,\partial^{-h} s^h, v)\,dt = -\int_0^{T-h} (\phi s^h, \partial^h v)\,dt$$
$$\to -\int_J (\phi s, \partial_t v)\,dt = \int_J \langle \phi\,\partial_t s, v\rangle\,dt,$$

as a distribution. Therefore, $\phi\,\partial^{-h} s^h \rightharpoonup \phi\,\partial_t s$ weakly in $L^2(J; W^*)$. Combining these results, the saturation equation holds in the weak sense of Definition 2.1 since $\bigcup_{M=1}^{\infty} l_h(W)$ is dense in $L^2(J; W)$.

Finally, if $v\in L^2(J; W)\cap W^{1,1}(J; L^1(\Omega))$ with $v(x,T)=0$, we find that

$$\int_J (\phi\,\partial^{-h} s^h, v)\,dt + \int_0^{T-h} (\phi[s^h - s^0], \partial^h v)\,dt = \frac{1}{h}\int_{T-h}^{T} (\phi[s^h - s^0], v)\,dt,$$

which yields the last equation in Definition 2.1, and thus the proof of Theorem 2.1 is complete. ∎

2.4. *Proof of Lemma* 2.4. In this subsection we finish the proof of Lemma 2.4. The next lemma is a simpler version of Lemma 3 in [6], and needed for proving Lemma 2.4.

LEMMA 2.5. *Let s^h satisfy* (2.16). *Then there exists C such that, for any $\zeta > 0$,*

$$\frac{1}{\zeta}\int_{\zeta}^{T} (\phi\{s^h(\cdot, t) - s^h(\cdot, t-\zeta)\}, \theta^h(\cdot, t) - \theta^h(\cdot, t-\zeta))\, dt \leqslant C.$$

Proof. Let k be fixed ($1 \leqslant k \leqslant m$); for $\tau \in J_i$, we define the interval $Q = Q(\tau) = ((i-k)\, h, ih]$, and the characteristic function χ_Q. Take $v(x, t) = kh\chi_Q(t)\, \partial^{-kh}(\theta^h - \varphi_4^h)(x, \tau) \in l_h(W)$ in (2.16), and apply the relation

$$\int_J \partial^{-h} s^h \chi_Q\, dt = \sum_{j=i-k+1}^{i} (s^j - s^{j-1}) = kh\, \partial^{-kh} s^h(\cdot, \tau),$$

(2.17), and (2.18) to obtain

$$\begin{aligned} kh \int_{kh}^{T} &(\phi\, \partial^{-kh} s^h(\cdot, \tau), \partial^{-kh}\theta^h(\cdot, \tau))\, d\tau \\ &\leqslant C + kh \int_{kh}^{T} (\phi\, \partial^{-kh} s^h(\cdot, \tau), \partial^{-kh}\varphi_4^h(\cdot, \tau))\, d\tau. \end{aligned}$$

As for (2.25), it can be shown that

$$\int_{kh}^{T} \|\partial^{-kh}\varphi_4^h\|_{L^1(\Omega)}\, d\tau \leqslant \|\partial_t \varphi_4\|_{L^1(\Omega_T)}.$$

Now, combine these two results to have the desired result since θ^h is constant on each interval J_i. ∎

For m_1 a positive integer, let $\delta = T/m_1$ and $J_k^\delta = ((k-1)\, \delta, k\delta]$. Following [6], introduce the operator $A^\delta\colon L^1(J) \to L^1(J)$ by

$$A^\delta(v) = \frac{1}{\delta}\int_{J_\kappa^\delta} v(\tau)\, d\tau, \qquad t \in J_k^\delta.$$

We are now in a position to prove Lemma 2.4.

Proof of Lemma 2.4. For any ζ, $N > 0$, define

$$\begin{aligned} Q = Q(\theta^h, \zeta, N) = \Big\{ t \in (\zeta, T] : &\|\theta^h(\cdot, t)\|_{H^1(\Omega)}^2 + \|\theta^h(\cdot, t-\zeta)\|_{H^1(\Omega)}^2 \\ &+ \frac{1}{\zeta}(\phi\{s^h(\cdot, t) - s^h(\cdot, t-\zeta)\}, \theta^h(\cdot, t) - \theta^h(\cdot, t-\zeta)) > N \Big\}. \end{aligned}$$

Obviously, by (2.17), (2.18), and Lemma 2.6, the measure of Q is less than C/N with constant C independent of h. If $t \in (\zeta, T]\backslash Q$, then

$$\|\theta^h(\cdot, t) - \theta^h(\cdot, t-\zeta)\|^2_{L^2(\Omega)} \leqslant \omega_N\left(\frac{\zeta N}{\phi_*}\right),$$

where $\omega_N\colon \mathscr{R} \to \mathscr{R}$ is a continuous function with $\omega_N(0) = 0$. Thus we see that

$$\int_\zeta^T \|\theta^h(\cdot, t) - \theta^h(\cdot, t-\zeta)\|^2_{L^2(\Omega)}\, dt \leqslant T\omega_N\left(\frac{\zeta N}{\phi_*}\right) + \frac{4C(\theta^*)\,|\Omega|}{N},$$

so that, by the arbitrariness of N,

$$\int_\zeta^T \|\theta^h(\cdot, t) - \theta^h(\cdot, t-\zeta)\|^2_{L^2(\Omega)}\, dt \to 0 \qquad \text{as} \quad \zeta \to 0^+,$$

uniformly in h. Therefore, by the definition of A^δ we see that

$$\int_J \|\theta^h - A^\delta(\theta^h)\|^2_{L^2(\Omega)}\, dt \leqslant \frac{2}{\delta} \int_0^\delta \int_\zeta^T \|\theta^h(\cdot, t) - \theta^h(\cdot, t-\zeta)\|^2_{L^2(\Omega)}\, dt\, d\zeta$$
$$\to 0 \qquad \text{as} \quad \delta \to 0^+, \tag{2.29}$$

uniformly in h. Also, $\|A^\delta(\theta^h)\|_{L^2(J,\, H^1(\Omega))}$ is uniformly bounded, so for fixed $\delta > 0$, $A^\delta(c^h)$ converges strongly in $L^2(\Omega_T)$ as $h \to 0^+$. Consequently, apply (2.29) and the inequality

$$\|\theta^{h_1} - \theta^{h_2}\|_{L^2(\Omega_T)} \leqslant \sum_{j=1}^{2} \|\theta^{h_j} - A^\delta(\theta^{h_j})\|_{L^2(\Omega_T)} + \|A^\delta(\theta^{h_1}) - A^\delta(\theta^{h_2})\|_{L^2(\Omega_T)},$$

to complete the proof of Lemma 2.4. ∎

3. UNIQUENESS OF THE WEAK SOLUTION

In this section we shall prove that the weak solution obtained in the last section is unique. For this, in addition to the assumptions (A1)–(A10), we also assume that

(A11) The following bound holds:

$$|\lambda(s_2) - \lambda(s_1)| + |\lambda_w(s_2) - \lambda_w(s_1)| + |\gamma_1(s_2) - \gamma_1(s_1)| + |\gamma_2(s_2) - \gamma_2(s_1)|$$
$$\leqslant C\sqrt{(\theta_2 - \theta_1)(s_2 - s_1)}, \qquad \forall 0 \leqslant \theta_1, \theta_2 \leqslant \theta^*(x), \tag{3.1}$$

where $s_i = \mathscr{S}(\theta_i)$, $i = 1, 2$.

Obviously, if the inequality holds

$$|\lambda_s| + |(\lambda_w)_s| + |(\gamma_1)_s| + |(\gamma_2)_s| \leqslant C \sqrt{\mathscr{S}_s^{-1}}, \qquad s \in [0, 1],$$

so does (3.1). For simplicity, we focus on the case where $\Gamma = \Gamma_2^p = \Gamma_2^\theta$, i.e., the Dirichlet problem; however, we point out that this restriction is not essential, as seen in the argument below. Also, we assume that q and q_w are independent of s; in the case where they depend on s, the following argument still works provided both q and q_w satisfy (3.1). The Lipschitz continuity (3.2) below on the pressure p will be shown in the next section. We now prove the main result in this section; the argument below follows an idea used in [19] for handling free-boundary problems.

THEOREM 3.1. *If, in addition to assumptions* (A1)–(A11),

$$|p(x_1, t) - p(x_2, t)| \leqslant C\, |x_1 - x_2|, \qquad x_1, x_2 \in \Omega,\ t \in J, \tag{3.2}$$

then the weak solution obtained in Theorem 2.1 *is unique.*

Proof. Let (p_1, θ_1) and (p_2, θ_2) be two solutions of the system in (2.8)–(2.14) in the sense of Definition 2.1 with $s_i = \mathscr{S}(\theta_i)$ $(i = 1, 2)$. Then, for $v \in L^\infty(J; V)$ and $z \in L^2(J; W) \cap W^{1,1}(J; L^1(\Omega))$ with $z(x, T) = 0$ it follows from Definition 2.1 and some manipulations that

$$\begin{aligned}\int_{\Omega_T} \{&(p_2 - p_1)\, \nabla \cdot [\kappa(\lambda(s_1)\, \nabla v + \lambda_w(s_1)\, \nabla z)] \\ &+ \phi(s_2 - s_1)[\partial_t z + e\nabla \cdot (\kappa \nabla z) + \sqrt{e}\, g_1 \cdot \nabla v + \sqrt{e}\, g_2 \cdot \nabla z]\}\, \mathrm{d}x\, \mathrm{d}t = 0,\end{aligned} \tag{3.3}$$

where

$$e = \begin{cases} (\theta_1 - \theta_2)/(\phi(s_1 - s_2)) & \text{if} \quad s_1 \neq s_2, \\ 0 & \text{if} \quad s_1 = s_2, \end{cases}$$

$$g_1 = \begin{cases} -\frac{\sqrt{e}\,\kappa}{\theta_1 - \theta_2} \{(\lambda(s_1) - \lambda(s_2))\, \nabla p_2 + (\gamma_1(s_1) - \gamma_1(s_2))\} & \text{if} \quad \theta_1 \neq \theta_2, \\ 0 & \text{if} \quad \theta_1 = \theta_2, \end{cases}$$

$$g_2 = \begin{cases} -\frac{\sqrt{e}\,\kappa}{\theta_1 - \theta_2} \{(\lambda_w(s_1) - \lambda_w(s_2))\, \nabla p_2 + (\gamma_2(s_1) - \gamma_2(s_2))\} & \text{if} \quad \theta_1 \neq \theta_2, \\ 0 & \text{if} \quad \theta_1 = \theta_2. \end{cases}$$

Note that $0 \leqslant e \leqslant C_1$, C_1 constant. Also, by (A11) and (3.2), we see that g_1 and g_2 are bounded functions on Ω_T.

Choose a sequence of functions $\tilde{e}_m$ infinitely differentiable in Ω_T such that

$$\tilde{e}_m \geqslant 0, \qquad \|\tilde{e}_m - e\|_{L^2(\Omega_T)} \leqslant \frac{1}{m}, \qquad \tilde{e}_m \leqslant 1 + C_1.$$

Set $e_m = \tilde{e}_m + 1/m$, so that

$$0 < e_m \leqslant 2 + C_1, \quad \|e/e_m\|_{L^2(\Omega_T)} \leqslant C, \quad \|e_m - e\|_{L^2(\Omega_T)} \to 0 \quad \text{as} \quad m \to \infty. \tag{3.4}$$

We also approximate g_1 and g_2 in $L^2(\Omega_T)$ by smooth g_1^m and g_2^m, respectively.

Suppose that $F(x, t)$ and $G(x, t)$ are arbitrary functions that are infinitely differentiable with compact support in Ω_T. We now define the pair of functions $(v_m, z_m) \in L^2(J; H_0^1(\Omega)) \times L^2(J; H_0^1(\Omega))$ to be the solution of the system

$$\begin{aligned} &\nabla \cdot \{\kappa(\lambda(s_1) \nabla v_m + \lambda_w(s_1) \nabla z_m)\} = F, \\ &\partial_t z_m + e_m \nabla \cdot (\kappa \nabla z_m) + \sqrt{e_m}\, g_1^m \cdot \nabla v_m + \sqrt{e_m}\, g_2^m \cdot \nabla z_m = G, \\ &z_m(x, T) = 0, \qquad x \in \Omega. \end{aligned} \tag{3.5}$$

The first equation of (3.5) is understood in the weak sense

$$(\kappa\{\lambda(s_1) \nabla v_m + \lambda_w(s_1) \nabla z_m\}, \nabla \psi) = -(F, \psi), \qquad \forall \psi \in L^\infty(J; H_0^1(\Omega)), \tag{3.6}$$

while the second equation is understood to hold almost everywhere in Ω_T. Existence and uniqueness of the linear system (3.5) can be seen from the a priori estimates derived below. First, it follows from (3.6) that

$$\|\nabla v_m\|_{L^2(\Omega)} \leqslant C(\|\nabla z_m\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}), \qquad t \in J. \tag{3.7}$$

Next, multiply the second equation of (3.5) by $\nabla \cdot (\kappa \nabla z_m)$, integrate the resulting equation over $\Omega \times (t, T)$, and use (3.7) and the Grownwall inequality to see that

$$\begin{aligned} &\|\nabla z_m(t)\|_{L^2(\Omega)}^2 + \int_t^T e_m (\nabla \cdot (\kappa \nabla z_m))^2 \, d\tau \\ &\qquad \leqslant C \int_t^T \int_\Omega \left(F^2 + \frac{G^2}{e_m}\right) dx \, d\tau, \qquad t \in J. \end{aligned} \tag{3.8}$$

We now estimate

$$I \equiv \int_{\Omega_T} \{(p_2 - p_1)\, F + \phi(s_2 - s_1)\, G\}\, \mathrm{d}x\, \mathrm{d}t.$$

If we take $v = v_m$ and $z = z_m$ in (3.3), we find that

$$I = \int_{\Omega_T} \phi(s_2 - s_1)\{(e_m - e)\, \nabla \cdot (\kappa \nabla z_m) + (\sqrt{e_m}\, g_1^m - \sqrt{e}\, g_1) \cdot \nabla v_m$$
$$+ (\sqrt{e_m}\, g_2^m - \sqrt{e}\, g_2) \cdot \nabla z_m\}\, \mathrm{d}x\, \mathrm{d}t. \tag{3.9}$$

Observe that

$$\int_{\Omega_T} |\phi(s_2 - s_1)(e_m - e)\, \nabla \cdot (\kappa \nabla z_m)|\, \mathrm{d}x\, \mathrm{d}t$$
$$\leqslant C \int_{\Omega_T} |e_m - e|\, |\nabla \cdot (\kappa \nabla z_m)|\, \mathrm{d}x\, \mathrm{d}t$$
$$\leqslant C \left\{ \int_{\Omega_T} \sqrt{e_m}\, |\sqrt{e_m} - \sqrt{e}|\, |\nabla \cdot (\kappa \nabla z_m)|\, \mathrm{d}x\, \mathrm{d}t \right.$$
$$\left. + \int_{\Omega_T} \sqrt{e}\, |\sqrt{e_m} - \sqrt{e}|\, |\nabla \cdot (\kappa \nabla z_m)|\, \mathrm{d}x\, \mathrm{d}t \right\}.$$

For any small $\eta > 0$, let

$$E_\eta = \Omega_T \cap \{|\sqrt{e_m} - \sqrt{e}| > \eta\}.$$

By (3.4), for any $\zeta > 0$ there is $m_1 = m_1(\eta, \zeta)$ such that $|E_\eta| < \zeta$ for all $m > m_1$. If $F_\eta = \Omega_T \backslash E_\eta$, we see that

$$\int_{\Omega_T} \sqrt{e_m}\, |\sqrt{e_m} - \sqrt{e}|\, |\nabla \cdot (\kappa \nabla z_m)|\, \mathrm{d}x\, \mathrm{d}t$$
$$\leqslant C \left\{ \eta \int_{F_\eta} \sqrt{e_m}\, |\nabla \cdot (\kappa \nabla z_m)|\, \mathrm{d}x\, \mathrm{d}t + \int_{E_\eta} \sqrt{e_m}\, |\nabla \cdot (\kappa \nabla z_m)|\, \mathrm{d}x\, \mathrm{d}t \right\}$$
$$\leqslant C(\eta + \sqrt{\zeta}) \left\{ \int_{\Omega_T} e_m (\nabla \cdot (\kappa \nabla z_m))^2\, \mathrm{d}x\, \mathrm{d}t \right\}^{1/2}.$$

Analogously, by (3.4), we have

$$\int_{\Omega_T} \sqrt{e}\,|\sqrt{e_m}-\sqrt{e}|\;|\nabla\cdot(\kappa\nabla z_m)|\;\mathrm{d}x\;\mathrm{d}t$$

$$\leqslant C(\eta+\zeta^{1/4})\left\{\int_{\Omega_T} e_m(\nabla\cdot(\kappa\nabla z_m))^2\;\mathrm{d}x\;\mathrm{d}t\right\}^{1/2}.$$

Applying (3.8), we thus obtain

$$\int_{\Omega_T} |\phi(s_2-s_1)(e_m-e)\;\nabla\cdot(\kappa\nabla z_m)|\;\mathrm{d}x\;\mathrm{d}t$$

$$\leqslant C(\eta+\zeta^{1/4})\left\{\int_{\Omega_T} (F^2+G^2/e_m)\;\mathrm{d}x\;\mathrm{d}t\right\}^{1/2}. \tag{3.10}$$

After choosing sequences of functions F_j and G_j that are infinitely differentiable with compact support in Ω_T and converge to p_2-p_1 and $\theta_2-\theta_1$ in $L^2(\Omega_T)$, respectively, we also have (3.10) for $F=p_2-p_1$ and $G=\theta_2-\theta_1$. Therefore, combine (3.9) and (3.10) to see that

$$\int_{\Omega_T} \{(p_2-p_1)^2+\phi(s_2-s_1)(\theta_2-\theta_1)\}\;\mathrm{d}x\;\mathrm{d}t$$

$$\leqslant C\,\{\eta+\zeta^{1/4}+\|g_1^m-g_1\|_{L^2(\Omega_T)}+\|g_2^m-g_2\|_{L^2(\Omega_T)}$$

$$+\|\sqrt{e_m}-\sqrt{e}\|_{L^2(\Omega_T)}\}\left\{\int_{\Omega_T}\{(p_2-p_1)^2+(\theta_2-\theta_1)^2/e_m\}\;\mathrm{d}x\;\mathrm{d}t\right\}^{1/2}. \tag{3.11}$$

From Lemma 3.1 below, as $m\to\infty$ we have

$$\int_{\Omega_T}\frac{(\theta_2-\theta_1)^2}{e_m}\;\mathrm{d}x\;\mathrm{d}t\to\int_{\Omega_T}\frac{(\theta_2-\theta_1)^2}{e}\;\mathrm{d}x\;\mathrm{d}t=\int_{\Omega_T}\phi(s_2-s_1)(\theta_2-\theta_1)\;\mathrm{d}x\;\mathrm{d}t,$$

which, together with (3.11) and the arbitrariness of η and ζ, implies that

$$\int_{\Omega_T}\{(p_2-p_1)^2+\phi(s_2-s_1)(\theta_2-\theta_1)\}\;\mathrm{d}x\;\mathrm{d}t=0,$$

so, $p_1=p_2$, $s_1=s_2$, and $\theta_1=\theta_2$. ∎

LEMMA 3.1. *If $G=\theta_2-\theta_1$ and $G^2/e=0$ by definition whenever $\theta_2=\theta_1$, then*

$$\int_{\Omega_T}\frac{G^2}{e_m}\,dx\,dt\to\int_{\Omega_T}\frac{G^2}{e}\,dx\,dt,\qquad as\quad m\to\infty.$$

This lemma can be shown as in Lemma 10.2 in [19] from the definition of e and the property of $s=\mathscr{S}(\theta)$.

4. REGULARITY OF THE WEAK SOLUTION

In the final section we shall consider the regularity of the weak solution obtained in the last two sections. To maintain consistency with the previous section, we only handle the case where $\Gamma=\Gamma_2^p=\Gamma_2^\theta$ and q and q_w are independent of s; the following results also hold for other boundary conditions (see [24] for extensions of results on problems of the Dirichlet type to other types).

4.1. *Regularity on the Pressure.* We first consider the pressure equation. Note that this equation is a standard elliptic problem. Thus the regularity theory on elliptic problems applies to it. However, since the coefficients in this equation depend upon the saturation s, the regularity on the pressure p is limited. We here summarize the regularity results applicable to the pressure equation.

The Hölder continuity on p below needs the assumption

(A12) The following norms are bounded,

$$\|q\|_{L^\infty(J;\,L^{r/2}(\Omega))},\qquad \|\gamma_1\|_{L^\infty(J;\,L^r(\Omega))},$$
$$\|\varphi_2\|_{L^\infty(J;\,C^\beta(\Gamma))},\qquad r>d,\qquad 0<\beta<1,$$

where we recall that d is the space dimension. We also assume that the domain Ω satisfies a uniform exterior cone condition on Γ (see [20] for the definition).

Under this assumption, we have

THEOREM 4.1. *If assumptions* (A1)–(A10) *and* (A12) *are satisfied, then*

$$\|p\|_{L^\infty(J;\,C^\beta(\Omega))}\leqslant C\big(\|p\|_{L^\infty(J;\,L^1(\Omega))}+\|q\|_{L^\infty(J;\,L^{r/2}(\Omega))}+\|\gamma_1\|_{L^\infty(J;\,L^r(\Omega))}+\|\varphi_2\|_{L^\infty(J;\,C^\beta(\Gamma))}\big).$$

The proof of this theorem can be carried out as in [20, 24]. Theorem 4.2 below will be needed in the proof of the Hölder continuity on s. Let $\kappa\lambda(s)\equiv\kappa_0(x)\,\lambda_0(x,t)$ and assume that

(A13) κ_0 is continuous on $\bar{\Omega}$ and λ_0 is measurable on Ω_T such that

$$|\lambda_0(x, t) - 1| \leqslant \delta < 1 \qquad a.e. \quad \text{on } \Omega_T. \tag{4.1}$$

If we refer to the graphs of λ_w and λ_n [9], we can see that (4.1) is reasonable. We remark that Theorem 4.2 below also holds if $\kappa\lambda(s) = (\kappa\lambda)(x, t)$ is continuous on Ω_T (instead of (4.1)). This would then require the continuity of s on Ω_T, which has not been proven yet; it was shown in [2, 5], though, under some conditions on the data. When $\kappa\lambda$ is continuous on $\bar{\Omega}_T$, the r_0 in Theorem 4.2 can be randomly taken in $(2, r]$.

(A14) We suppose that $\|\varphi_2\|_{L^\infty(J;\, W^{1,r}(\Omega))}$ is bounded with $r > d$.

Below $\Lambda_y = \Lambda(y, \Omega, d)$ is a continuous function of y with $\Lambda_2 \leqslant 1$.

THEOREM 4.2. *Under assumptions* (A1)–(A10) *and* (A12)–(A14), *we have*

$$\|p\|_{L^\infty(J;\, W^{1,r_0}(\Omega))} \leqslant C(\|q\|_{L^\infty(J;\, L^{r/2}(\Omega))} + \||\gamma_1\||_{L^\infty(J;\, L^r(\Omega))} + \|\varphi_2\|_{L^\infty(J;\, W^{1,r}(\Omega))})^{r_0},$$

where $2 < r_0 \leqslant r$ *satisfies that* $\delta\Lambda_{r_0} < 1$.

The proof of this theorem can be found in [5], for example.

4.2. *Regularity on the Saturation.* We now consider a regularity result on the saturation. For this, we introduce some notation. For $v \in L^1(\Omega_T)$ and $0 < h < T$, introduce the Steklov average v_h by

$$v_h(x, t) = \begin{cases} \dfrac{1}{h} \displaystyle\int_t^{t+h} v(x, \tau)\, d\tau & \text{if } \ 0 < t \leqslant T - h, \\ 0 & \text{if } \ t > T - h. \end{cases}$$

Then it follows from Definition 2.1 that the following equation is equivalent to the saturation equation,

$$(\phi\, \partial_t s_h, v)_K + ((\kappa \nabla \theta)_h + (\kappa \lambda_w(s)\, \nabla p)_h + (\kappa \gamma_2(s))_h, \nabla v)_K = ((q_w)_h, v)_K, \qquad 0 < t < T - h, \qquad \forall v \in H_0^1(K) \cap L_{loc}^\infty(\Omega), \tag{4.2}$$

for any compact subset K of Ω.

The next lemma follows easily from the theory of $L^p(\Omega)$ spaces [24].

LEMMA 4.1. *Let* $r_1, \pi \geqslant 1$. *If* $v \in L^{r_1}(J; L^\pi(\Omega))$, *then for all* $\varepsilon \in (0, T)$ $v_h \to v$ *in* $L^{r_1}(0, T - \varepsilon; L^\pi(\Omega))$ *as* $h \to 0^+$. *Also, if* $v \in C(J; L^\pi(\Omega))$, *then for all* $\varepsilon \in (0, T)$ $v_h(\cdot, t) \to v(\cdot, t)$ *in* $L^\pi(\Omega)$ *as* $h \to 0^+$ *for every* $t \in (0, T - \varepsilon)$.

We shall prove two integral inequalities in the interior of Ω_T, which utilize the following notation. For $\rho > 0$, define the cube

$$K_\rho = \{x \in \mathscr{R}^d \colon \max_{1 \leqslant i \leqslant d} |x_i| < \rho\}.$$

For $x_0 \in \mathscr{R}^d$, let $x_0 + K_\rho$ denote the cube of center x_0. Also, for $\eta > 0$ a given number, define

$$Q(\eta, \rho) = K_\rho \times (-\eta, 0).$$

For $(x_0, t_0) \in \mathscr{R}^{d+1}$, let $(x_0, t_0) + Q(\eta, \rho)$ be the cylinder congruent to $Q(\eta, \rho)$; i.e.,

$$(x_0, t_0) + Q(\eta, \rho) = \{x_0 + K_\rho\} \times (t_0 - \eta, t_0).$$

For $(x_0, t_0) \in \Omega_T$ fixed, let ρ and η be so small that $(x_0, t_0) + Q(\eta, \rho) \subset \Omega_T$. In $(x_0, t_0) + Q(\eta, \rho)$, introduce piecewise smooth cut-off functions $\xi(x, t)$ and $\zeta(x)$ such that they satisfy

$$\xi \in [0, 1], \qquad |\nabla \xi| < \infty, \qquad \xi(x, t) = 0 \qquad \text{for} \quad x \notin x_0 + K_\rho. \tag{4.3}$$

Let l be any real number, and define the truncations of the saturation s by

$$(s-l)_+ = \max\{s-l, 0\}, \qquad (s-l)_- = \max\{-(s-l), 0\}.$$

Also, we choose levels l such that

$$H_l^\pm \equiv \operatorname*{ess\,sup}_{(x_0,\, t_0) + Q(\eta, \rho)} |(s-l)_\pm| \leqslant \delta_1, \tag{4.4}$$

where δ_1 is a positive parameter to be determined later. Finally, set

$$A_{l,\rho}^\pm(\tau) = \{x \in x_0 + K_\rho : (s(x, \tau) - l)_\pm > 0\},$$

and

$$\Psi(H_l^\pm, (s-l)_\pm, v) = \ln^+ \left\{ \frac{H_l^\pm}{H_l^\pm - (s-l)_\pm + v} \right\}, \qquad 0 < v < \min\{H_l^\pm, 1\},$$

where $\ln^+ v = \max\{\ln v, 0\}$.

We are now ready to prove the next two lemmas. For this, assume that

(A15) The norm $\|q_w\|_{L^2(J;\, L^{r_0/2}(\Omega))}$ is bounded, where r_0 is from Theorem 4.2, and

$$|\lambda_w(s)| + |\gamma_2(s)| \leqslant C \sqrt{\mathscr{S}_s^{-1}(s)}, \qquad s \in [0, 1].$$

Remark 4.1. Under this additional condition, it can be shown as in Lemma 2.3 that $\sqrt{\mathcal{S}_s^{-1}}\, \nabla s \in L^2(\Omega_T)$. Also, note that (4.2) holds with any function v such that $\sqrt{\mathcal{S}_s^{-1}}\, \nabla v \in L^2(\Omega)$.

LEMMA 4.2. *In addition to assumptions of Theorem* 4.2, *let* (A15) *be satisfied. Then there are constants* C, C_1, *and* δ_1^* *depending only on the data such that for every cylinder* $(x_0, t_0) + Q(\eta, \rho) \subset \Omega_T$ *and every level* l *satisfying* (4.4) *for* $\delta_1 \leqslant \delta_1^*$

$$\sup_{t_0-\eta<t<t_0} \int_{x_0+K_\rho} (s-l)_\pm^2\, \xi^2\, \mathrm{d}x + C_1 \int_{(x_0,\,t_0)+Q(\eta,\,\rho)} \mathcal{S}_s^{-1}\, |\xi \nabla (s-l)_\pm|^2\, \mathrm{d}x\, \mathrm{d}\tau$$

$$\leqslant \int_{x_0+K_\rho} ((s-l)_\pm^2\, \xi^2)(x, t_0-\eta)\, \mathrm{d}x$$

$$+ C \int_{(x_0,\,t_0)+Q(\eta,\,\rho)} \mathcal{S}_s^{-1} (s-l)_\pm^2\, |\nabla \xi|^2\, \mathrm{d}x\, \mathrm{d}\tau$$

$$+ C \int_{(x_0,\,t_0)+Q(\eta,\,\rho)} (s-l)_\pm^2\, \xi\, \partial_t \xi\, \mathrm{d}x\, \mathrm{d}\tau$$

$$+ C(\delta_1^*) \left(\int_{t_0-\eta}^{t_0} |A_{l,\rho}^{\pm}(\tau)|^{2(1-2/r_0)}\, \mathrm{d}\tau \right)^{1/2},$$

with the piecewise smooth cut-off function ξ *satisfying* (4.3).

Proof. Without loss of generality, assume that $(x_0, t_0) = (0, 0)$. Take $v = \pm (s_h - l)_\pm \xi^2$ in (4.2) (which is admissible by Remark 4.1) and integrate the resulting equation over $(-\eta, t)$, $t \in (-\eta, 0)$, to see that

$$\int_{-\eta}^{t} (\phi\, \partial_t s_h,\, \pm (s_h - l)_\pm \xi^2)_{K_\rho}\, \mathrm{d}\tau$$

$$+ \int_{-\eta}^{t} ((\kappa \nabla \theta)_h + (\kappa \lambda_w(s)\, \nabla p)_h + (\kappa \gamma_2(s))_h,\, \pm \nabla [(s_h - l)_\pm \xi^2])_{K_\rho}\, \mathrm{d}\tau$$

$$= \int_{-\eta}^{t} ((q_w)_h,\, \pm (s_h - l)_\pm \xi^2)_{K_\rho}\, \mathrm{d}\tau. \tag{4.5}$$

We now estimate each of the terms in (4.5). First, note that

$$\int_{-\eta}^{t} (\phi\, \partial_t s_h,\, \pm (s_h - l)_\pm\, \xi^2)_{K_\rho}\, \mathrm{d}\tau = \frac{1}{2} \int_{-\eta}^{t} (\phi \partial_t (s_h - l)_\pm^2,\, \xi^2)_{K_\rho}\, \mathrm{d}\tau,$$

so, integrate by parts in t, let $h \to 0^+$, and apply Lemma 4.1 to see that

$$\int_{-\eta}^{t} (\phi\, \partial_t s_h,\, \pm(s_h - l)_\pm\, \xi^2)_{K_\rho}\, \mathrm{d}\tau \to \tfrac{1}{2}\,(\phi(s-l)_\pm^2\,(t),\, \xi^2(t))_{K_\rho}$$

$$- \tfrac{1}{2}\,(\phi(s-l)_\pm^2\,(-\eta),\, \xi^2(-\eta))_{K_\rho}$$

$$- \int_{-\eta}^{t} (\phi(s-l)_\pm^2,\, \xi\partial_t\xi)_{K_\rho}\, \mathrm{d}\tau. \quad (4.6)$$

Next, letting $h \to 0^+$, observe that

$$\pm \int_{-\eta}^{t} ((\kappa\nabla\theta)_h,\, \nabla[(s_h - l)_\pm \xi^2])_{K_\rho}\, \mathrm{d}\tau$$

$$\to \pm \int_{-\eta}^{t} (\kappa\nabla\theta,\, \xi^2\nabla(s-l)_\pm + 2(s-l)_\pm\, \xi\nabla\xi)_{K_\rho}\, \mathrm{d}\tau$$

$$= \pm \int_{-\eta}^{t} (\kappa\mathscr{S}_s^{-1}\nabla s,\, \xi^2\nabla(s-l)_\pm + 2(s-l)_\pm\, \xi\nabla\xi)_{K_\rho}\, \mathrm{d}\tau$$

$$\geqslant C_1 \int_{-\eta}^{t} \|\sqrt{\mathscr{S}_s^{-1}}\, \xi\nabla(s-l)_\pm\|_{L^2(K_\rho)}^2\, \mathrm{d}\tau$$

$$- C \int_{-\eta}^{t} \|\sqrt{\mathscr{S}_s^{-1}}(s-l)_\pm\, \nabla\xi\|_{L^2(K_\rho)}^2\, \mathrm{d}\tau. \quad (4.7)$$

Similarly, by (4.3) and (A15), we see that

$$\pm \int_{-\eta}^{t} ((\kappa\lambda_w(s)\, \nabla p)_h,\, \nabla[(s_h - l)_\pm\, \xi^2])_{K_\rho}\, \mathrm{d}\tau$$

$$\to \pm \int_{-\eta}^{t} (\kappa\lambda_w(s)\, \nabla p,\, \xi^2\nabla(s-l)_\pm + 2(s-l)_\pm\, \xi\nabla\xi)_{K_\rho}\, \mathrm{d}\tau$$

$$\leqslant \frac{C_1}{4} \int_{-\eta}^{t} \|\sqrt{\mathscr{S}_s^{-1}}\, \xi\nabla(s-l)_\pm\|_{L^2(K_\rho)}^2\, \mathrm{d}\tau$$

$$+ C \int_{-\eta}^{t} \|\sqrt{\mathscr{S}_s^{-1}}\, (s-l)_\pm\, \nabla\xi\|_{L^2(K_\rho)}^2\, \mathrm{d}\tau$$

$$+ C \int_{-\eta}^{t} (|\nabla p|^2,\, \chi[(s-l)_\pm > 0])_{K_\rho}\, \mathrm{d}\tau, \quad (4.8)$$

and

$$\pm\int_{-\eta}^{t}((\kappa\gamma_2(s))_h, \nabla[(s_h-l)_\pm\,\xi^2])_{K_\rho}\,d\tau$$

$$\to \pm\int_{-\eta}^{t}(\kappa\gamma_2(s), \xi^2\nabla(s-l)_\pm + 2(s-l)_\pm\,\xi\nabla\xi)_{K_\rho}\,d\tau$$

$$\leqslant \frac{C_1}{4}\int_{-\eta}^{t}\|\sqrt{\mathscr{S}_s^{-1}}\,\xi\nabla(s-l)_\pm\|^2_{L^2(K_\rho)}\,d\tau$$

$$+ C\int_{-\eta}^{t}\|\sqrt{\mathscr{S}_s^{-1}}\,(s-l)_\pm\,\nabla\xi\|^2_{L^2(K_\rho)}\,d\tau$$

$$+ C\int_{-\eta}^{t}(1, \chi[(s-l)_\pm > 0])_{K_\rho}\,d\tau, \tag{4.9}$$

where $\chi[(s-l)_\pm > 0]$ is the characteristic function of the set $\{(s-l)_\pm > 0\}$. Finally, we see that

$$\pm\int_{-\eta}^{t}((q_w)_h, (s_h-l)_\pm\,\xi^2)_{K_\rho}\,d\tau \to \pm\int_{-\eta}^{t}(q_w, (s-l)_\pm\,\xi^2)_{K_\rho}\,d\tau$$

$$\leqslant \delta_1\int_{-\eta}^{t}(|q_w|, \chi[(s-l)_\pm > 0])_{K_\rho}\,d\tau. \tag{4.10}$$

Now, combine (4.5)–(4.10) to obtain

$$(\phi(s-l)^2_\pm(t), \xi^2(t))_{K_\rho} + C_1\int_{-\eta}^{t}\|\sqrt{\mathscr{S}_s^{-1}}\,\xi\nabla(s-l)_\pm\|^2_{L^2(K_\rho)}\,d\tau$$

$$\leqslant (\phi(s-l)^2_\pm(-\eta), \xi^2(-\eta))_{K_\rho} + C\int_{-\eta}^{t}((s-l)^2_\pm, \xi\partial_t\xi)_{K_\rho}\,d\tau$$

$$+ C\int_{-\eta}^{t}\|\sqrt{\mathscr{S}_s^{-1}}\,(s-l)_\pm\,\nabla\xi\|^2_{L^2(K_\rho)}\,d\tau$$

$$+ C(\delta_1)\int_{-\eta}^{t}(1 + |\nabla p|^2 + |q_w|, \chi[(s-l)_\pm > 0])_{K_\rho}\,d\tau,$$

which, together with Hölder's inequality, implies the desired result. ∎

LEMMA 4.3. *Under the assumptions of Lemma* 4.2, *there is constant* C *depending only on the data such that for every cylinder* $(x_0, t_0) + Q(\eta, \rho) \subset \Omega_T$ *and every level* l *satisfying* (4.4) for all δ_1

$$\sup_{t_0-\eta<t<t_0} \int_{x_0+K_\rho} \Psi^2(H_l^\pm, (s-l)_\pm, \nu)(x,t)\,\zeta^2(x)\,\mathrm{d}x$$

$$\leqslant \int_{x_0+K_\rho} \Psi^2(H_l^\pm, (s-l)_\pm, \nu)(x, t_0-\eta)\,\zeta^2(x)\,\mathrm{d}x$$

$$+ C\int_{(x_0,t_0)+Q(\eta,\rho)} \mathcal{S}_s^{-1}\,\Psi(H_l^\pm, (s-l)_\pm, \nu)\,|\nabla\zeta|^2\,\mathrm{d}x\,\mathrm{d}\tau$$

$$+\frac{C}{\nu^2}\left(1+\ln\frac{H_l^\pm}{\nu}\right)\left(\int_{t_0-\eta}^{t_0} |A_{l,\rho}^\pm(\tau)|^{2(1-2/r_0)}\,\mathrm{d}\tau\right)^{1/2},$$

with the piecewise smooth cut-off function ζ *satisfying* (4.3).

Proof. As in the proof of Lemma 4.2, let $(x_0, t_0)=(0,0)$. Also, for notational convenience, set

$$\psi(s) \equiv \Psi(H_l^\pm, (s-l)_\pm, \nu).$$

Since $(\psi^2(s_h))'' = 2(1+\psi)(\psi')^2 \in L^\infty_{loc}(\Omega_T)$ by the definition of Ψ, we can take $v=(\psi^2(s_h))'\,\zeta^2$ in (4.2) by Remark 4.1 to see that

$$\int_{-\eta}^{t} (\phi\,\partial_t s_h, (\psi^2)'\,\zeta^2)_{K_\rho}\,\mathrm{d}\tau$$

$$+\int_{-\eta}^{t} ((\kappa\nabla\theta)_h + (\kappa\lambda_w(s)\,\nabla p)_h + (\kappa\gamma_2(s))_h, \nabla[(\psi^2)'\,\zeta^2])_{K_\rho}\,\mathrm{d}\tau$$

$$=\int_{-\eta}^{t} ((q_w)_h, (\psi^2)'\,\zeta^2)_{K_\rho}\,\mathrm{d}\tau. \tag{4.11}$$

Each of the terms in (4.11) is estimated similarly as for (4.5). First, we observe that

$$\int_{-\eta}^{t} (\phi\partial_t s_h, (\psi^2)'\,\zeta^2)_{K_\rho}\,\mathrm{d}\tau = \int_{-\eta}^{t} (\phi\partial_t\psi^2, \zeta^2)_{K_\rho}\,\mathrm{d}\tau$$

$$= (\phi\psi^2(s_h)(t), \zeta^2)_{K_\rho} - (\phi\psi^2(s_h)(-\eta), \zeta^2)_{K_\rho},$$

so, letting $h\to 0^+$,

$$\int_{-\eta}^{t} (\phi\,\partial_t s_h, (\psi^2)'\,\zeta^2)_{K_\rho}\,\mathrm{d}\tau \to (\phi\psi^2(s)(t), \zeta^2)_{K_\rho} - (\phi\psi^2(s)(-\eta), \zeta^2)_{K_\rho}.$$

Also, we have

$$\int_{-\eta}^{t} ((\kappa\nabla\theta)_h, \nabla[(\psi^2)'\,\zeta^2])_{K_\rho}\,\mathrm{d}\tau$$

$$\to \int_{-\eta}^{t} (\kappa\nabla\theta, \nabla[(\psi^2)'\,\zeta^2])_{K_\rho}\,\mathrm{d}\tau$$

$$\geqslant C_1 \int_{-\eta}^{t} (\mathscr{S}_s^{-1}\,\nabla s, (1+\psi)(\psi')^2\,\zeta^2\nabla s)_{K_\rho}\,\mathrm{d}\tau$$

$$- C \int_{-\eta}^{t} (\mathscr{S}_s^{-1}\,\nabla\zeta, \psi\,\nabla\zeta)_{K_\rho}\,\mathrm{d}\tau,$$

$$\int_{-\eta}^{t} ((\kappa\lambda_w(s)\,\nabla p)_h, \nabla[(\psi^2)'\,\zeta^2])_{K_\rho}\,\mathrm{d}\tau$$

$$\to \int_{-\eta}^{t} (\kappa\lambda_w(s)\,\nabla p, \nabla[(\psi^2)'\,\zeta^2])_{K_\rho}\,\mathrm{d}\tau$$

$$\leqslant \frac{C_1}{4} \int_{-\eta}^{t} (\mathscr{S}_s^{-1}\,\nabla s, (1+\psi)(\psi')^2\,\zeta^2\nabla s)_{K_\rho}\,\mathrm{d}\tau$$

$$+ C\left\{\int_{-\eta}^{t} (|\nabla p|^2, (1+\psi)(\psi')^2\,\zeta^2)_{K_\rho}\,\mathrm{d}\tau\right.$$

$$+ \int_{-\eta}^{t} (\mathscr{S}_s^{-1}\,\nabla\zeta, \psi\nabla\zeta)_{K_\rho}\,\mathrm{d}\tau$$

$$\left.+ \int_{-\eta}^{t} (|\nabla p|^2, \psi(\psi')^2\,\zeta^2)_{K_\rho}\,\mathrm{d}\tau\right\},$$

$$\int_{-\eta}^{t} ((\kappa\gamma_2(s))_h, \nabla[(\psi^2)'\,\zeta^2])_{K_\rho}\,\mathrm{d}\tau$$

$$\to \int_{-\eta}^{t} (\kappa\gamma_2(s), \nabla[(\psi^2)'\,\zeta^2])_{K_\rho}\,\mathrm{d}\tau$$

$$\leqslant \frac{C_1}{4} \int_{-\eta}^{t} (\mathscr{S}_s^{-1}\nabla s, (1+\psi)(\psi')^2\,\zeta^2\nabla s)_{K_\rho}\,\mathrm{d}\tau$$

$$+ C\left\{\int_{-\eta}^{t} (1, (1+\psi)(\psi')^2\,\zeta^2)_{K_\rho}\,\mathrm{d}\tau\right.$$

$$+ \int_{-\eta}^{t} (\mathscr{S}_s^{-1}\,\nabla\zeta, \psi\nabla\zeta)_{K_\rho}\,\mathrm{d}\tau$$

$$\left.+ \int_{-\eta}^{t} (1, \psi(\psi')^2\,\zeta^2)_{K_\rho}\,\mathrm{d}\tau\right\},$$

and

$$\int_{-\eta}^{t} ((q_w)_h, (\psi^2)' \zeta^2)_{K_\rho} \, d\tau \to \int_{-\eta}^{t} (q_w, (\psi^2)' \zeta^2)_{K_\rho} \, d\tau$$
$$\leqslant C \int_{-\eta}^{t} (|q_w|, \psi\psi'\zeta^2)_{K_\rho} \, d\tau.$$

Note that, by the definition of ψ and Ψ,

$$\psi \leqslant \ln \frac{H_l^{\pm}}{v}, \qquad \psi' \leqslant \frac{1}{v}, \qquad v < 1.$$

Therefore, combine all these estimates in (4.11) to see that

$$\begin{aligned}
&(\phi \Psi^2(H_l^{\pm}, (s-l)_{\pm}, v)(t), \zeta^2)_{K_\rho} \\
&\quad \leqslant (\phi \Psi^2(H_l^{\pm}, (s-l)_{\pm}, v)(-\eta), \zeta^2)_{K_\rho} \\
&\qquad + C \int_{-\eta}^{t} (\mathcal{S}_s^{-1} \nabla\zeta, \Psi(H_l^{\pm}, (s-l)_{\pm}, v) \nabla\zeta)_{K_\rho} \, d\tau \\
&\qquad + \frac{C}{v^2}\left(1 + \ln \frac{H_l^{\pm}}{v}\right) \int_{-\eta}^{t} (1 + |\nabla p|^2 + |q_w|, \chi[(s-l)_{\pm} > 0])_{K_\rho} \, d\tau,
\end{aligned}$$

which, again together with Hölder's inequality, implies the desired result. ∎

From Lemmas 4.2 and 4.3 the Hölder continuity of the saturation s can be shown using the standard technique [17, 18]. As an example, we apply the theory in [18] to deduce our result. However, to use the so-called alternative argument in [18], we have to make an assumption on the degeneracy of $\mathcal{S}_s^{-1}$ near the origin. In particular, we assume that

(A16) There is a number $\sigma_0 > 0$ such that

$$c_1 s^{\beta_1} \leqslant \mathcal{S}_s^{-1}(s) \leqslant c_2 s^{\beta_2}, \qquad 0 \leqslant s \leqslant \sigma_0, \tag{4.12}$$

for given positive constants $0 < c_1 \leqslant c_2$ and $0 \leqslant \beta_2 \leqslant \beta_1$. For $s > \sigma_0$, $\mathcal{S}_s^{-1}$ is bounded below and above:

$$0 < C_1 \leqslant \mathcal{S}_s^{-1}(s) \leqslant C_2, \qquad \sigma_0 < s \leqslant 1. \tag{4.13}$$

By symmetry, the following two theorems also hold by assuming (A16) near $s = 1$; i.e., inequality (4.12) is satisfied for $\sigma_0 \leqslant s \leqslant 1$ and (4.13) for $0 \leqslant s < \sigma_0$. The Hölder continuity of s without assumption (A16) is being investigated, which needs to extend the De Giorgi original technique [17].

THEOREM 4.3 (Local Regularity). *Under assumptions* (A1)–(A10) *and* (A12)–(A16), *s is locally Hölder continuous in* Ω_T; *i.e., for every compact subset K of* Ω_T, *there exist constants* $C>0$ *and* $\beta\in(0,1)$ *such that*

$$|s(x_1,t_1)-s(x_2,t_2)|\leqslant C(|x_1-x_2|^{\beta}+|t_1-t_2|^{\beta/2}),$$

for every pair of points (x_1,t_1), $(x_2,t_2)\in K$.

Remark 4.2. The proof of this theorem follows from Lemmas 4.2 and 4.3 and the alternative argument in [18]. Note that while a different kind of degeneracy is considered in [18] (i.e., the gradient degeneracy is considered there), the same argument applies here; see [26] for applying the alternative argument for the same degeneracy as that analyzed here. Also, note that the argument in [18, 26] applies only to a single degeneracy in $\mathcal{S}_s^{-1}(s)$. That is why we have assumed (A16). In practice, $\mathcal{S}_s^{-1}(s)$ has a dual degeneracy at zero and one. In this case, the standard alternative argument must be modified to reflect this dual degeneracy, which is being studied by the author.

What is critical here is the assumption that we have made in (A16).

To have the global continuity, we need the corresponding continuity assumption on the initial and boundary data.

(A17) Assume that $s_4=\mathcal{S}(\varphi_4)$ and $s_0=\mathcal{S}(\theta_0)$ are Hölder continuous on $\Gamma\times J$ and Ω, respectively.

THEOREM 4.4 (Global Regularity). *Under assumptions* (A1)–(A10) *and* (A12)–(A17), *s is Hölder continuous on* $\bar{\Omega}_T$; *i.e., there exist constants* $C>0$ *and* $\beta\in(0,1)$ *such that*

$$|s(x_1,t_1)-s(x_2,t_2)|\leqslant C(|x_1-x_2|^{\beta}+|t_1-t_2|^{\beta/2}),$$

for every pair of points (x_1,t_1), $(x_2,t_2)\in\bar{\Omega}_T$.

The proof of Theorem 4.4 follows from Theorem 4.3 and the usual argument in handling boundary regularity [18].

4.3. *Assumption* (3.2). We finally state a result which implies assumption (3.2). Toward that end, we need the following assumption:

(A18) The domain Ω is $C^{1,\beta}$ $(0<\beta<1)$, $\varphi_2\in L^\infty(J;C^{1,\beta}(\Omega))$, and $q\in L^\infty(J;L^2(\Omega))$.

THEOREM 4.5. *Let assumptions* (A1)–(A10) *and* (A12)–(A18) *be satisfied. Then the pressure p satisfies*

$$\|\nabla p\|_{L^\infty(J;C^\beta(\Omega))}\leqslant C(\|p\|_{L^\infty(J;L^2(\Omega))}+\|q\|_{L^\infty(J;L^2(\Omega))}$$
$$+\|\varphi_2\|_{L^\infty(J;C^{1,\beta}(\Omega))}+|||\gamma_1|||_{L^\infty(J;C^\beta(\Omega))}).$$

The proof of this theorem follows from Corollary 8.35 in [20]. It depends only on (A1)–(A10), (A18), and the Hölder continuity of s. As mentioned above, we have shown the latter continuity under assumption (A16), as an example, and it will be further studied under more general assumptions on the degeneracy of $\mathscr{S}_s^{-1}$.

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