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Degenerate two-phase incompressible flow II: regularity, stability and stabilization ☆

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Abstract

In this paper, we analyze a coupled system of highly degenerate elliptic-parabolic partial differential equations for two-phase incompressible flow in porous media. This system involves a saturation and a global pressure (or a total flow velocity). First, we show that the saturation is Hölder continuous both in space and time and the total velocity is Hölder continuous in space (uniformly in time). Applying this regularity result, we then establish the stability of the saturation and pressure with respect to initial and boundary data, from which uniqueness of the solution to the system follows. Finally, we establish a stabilization result on the asymptotic behavior of the saturation and pressure; we prove that the solution to the present system converges (in appropriate norms) to the solution of a stationary system as time goes to infinity. An example is given to show typical regularity of the saturation.

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1. Introduction

In this paper, we consider the flow of two incompressible, immiscible fluids in porous media $\Omega \subset \Re^d$, $d \leqslant 3$ [8,25]:

$$
\begin{aligned}
&\phi \partial_t s - \nabla \cdot (\kappa \lambda_w(s)(\nabla p_w + \gamma_w)) = 0,\\
&-\ \phi \partial_t s - \nabla \cdot (\kappa \lambda_0(s)(\nabla p_0 + \gamma_0)) = 0,\\
&p_c(s) = p_0 - p_w,
\end{aligned} \tag{1.1}
$$

where w indicates a wetting phase (e.g., water), o denotes a nonwetting phase (e.g., oil), ϕ and κ are the porosity and absolute permeability of the porous media, s is the (reduced) saturation of the wetting phase, p_α, λ_α, and γ_α, are, respectively, the pressure, mobility (i.e., the relative permeability over the viscosity), and gravity-density vector of the α-phase ($\alpha = w, o$), and p_c is the capillary pressure function. To analyze (1.1), following [2,9], we define the global pressure

$$
p = p_0 - \int_0^s \left(\frac{\lambda_w}{\lambda}\frac{\partial p_c}{\partial s}\right)(\xi)\, d\xi, \tag{1.2}
$$

and [10], the Kirchhoff transformation

$$
\theta = -\int_0^s \left(\frac{\lambda_w \lambda_0}{\lambda}\frac{\partial p_c}{\partial s}\right)(\xi)\, d\xi, \tag{1.3}
$$

where $\lambda(s) = \lambda_w + \lambda_0$ is the total mobility. Then (1.1) can be manipulated to yield the Eq. [10]

$$
u = -\kappa(\lambda(s)\nabla p + \gamma_1(s)), \quad \nabla \cdot u = 0, \tag{1.4}
$$

and

$$
\phi \partial_t s - \nabla \cdot \{\kappa(\nabla \theta + \gamma_2(s)) + u\gamma_3(s)\} = 0, \tag{1.5}
$$

where

$$
\begin{aligned}
&\gamma_1(s) = \lambda_w \gamma_w + \lambda_0 \gamma_0, \quad \gamma_2(s) = \frac{\lambda_w \lambda_0}{\lambda}(\gamma_w - \gamma_0),\\
&\quad \gamma_3(s) = -\frac{\lambda_w}{\lambda} \text{ or } \frac{\lambda_0}{\lambda}.
\end{aligned} \tag{1.6}
$$

In (1.5), s is related to θ through (1.3):

$$
s = \mathcal{S}(\theta), \tag{1.7}
$$

where $\mathcal{S}(\theta)$ is the inverse of (1.3) for $0 \leqslant \theta \leqslant \theta^*$ with

$$
\theta^* = -\int_0^1 \left(\frac{\lambda_w \lambda_0}{\lambda}\frac{\partial p_c}{\partial s}\right)(\xi)\, d\xi.
$$

The pressure equation is given by (1.4), while the saturation equation is described by (1.5). They determine the main unknowns p (or u the total flow velocity [10]), s, and θ. The model is completed by specifying boundary and initial conditions.

With the following division of the boundary Γ of Ω:

$$\Gamma = \Gamma_1^p \cup \Gamma_2^p = \Gamma_1^\theta \cup \Gamma_2^\theta, \quad \emptyset = \Gamma_1^p \cap \Gamma_2^p = \Gamma_1^\theta \cap \Gamma_2^\theta,$$

the boundary conditions are specified by

$$\begin{aligned} &u \cdot v = \varphi_1(x,t), && (x,t) \in \Gamma_1^p \times J, \\ &p = \varphi_2(x,t), && (x,t) \in \Gamma_2^p \times J, \\ &-\{\kappa(\nabla\theta + \gamma_2(s)) + u\gamma_3(s)\} \cdot v = \varphi_3(x,t), && (x,t) \in \Gamma_1^\theta \times J, \\ &\theta = \varphi_4(x,t), && (x,t) \in \Gamma_2^\theta \times J, \end{aligned} \tag{1.8}$$

where the φ_i are given functions, $J = (0, T]$ $(T > 0)$, and v is the outer unit normal to Γ. The initial condition is given by

$$s(x,0) = s_0(x), \quad x \in \Omega. \tag{1.9}$$

The differential system in (1.4) and (1.5) has a clear structure; the pressure equation is elliptic for p, and the saturation equation is parabolic for θ (degenerate for s). This system has been recently studied in [10]. In particular, existence of a weak solution (in the sense given in [10]) was established under reasonable assumptions on physical data (also see [1,3,4,9,21,22] for the existence under various assumptions on the data), and a regularity result on the Hölder continuity of the saturation s was obtained with the assumption that (1.5) has one degeneracy in diffusivity.

In this paper, we further study the coupled system of differential equations in (1.4) and (1.5). First, we show that the saturation s is Hölder continuous both in space and time and the total velocity u is Hölder continuous in space (uniformly in time). The assumptions imposed in [10] are weakened; physically reasonable assumptions on the data are used. Especially, (1.5) can have two degeneracies in diffusivity near zero and one. Due to the two degeneracies, the argument here is different from that in [10]; these two degeneracies have to be related to each other in the argument. Applying this regularity result, we then establish the stability of s and p with respect to initial and boundary data, from which uniqueness of the solution to this system follows. A uniqueness result was obtained in [10], where the uniqueness was directly proven, while it follows from the stability here. The arguments are different. Finally, we establish a stabilization result on the asymptotic behavior of s and p; we prove that the solution to the present system converges (in appropriate norms) to the solution of a stationary system as time progresses to infinity. This result corresponds to the physical case where the wetting phase completely displaces the nonwetting phase (which initially occupied the domain Ω) under the assumption that the residual saturations are zero.

The rest of the paper is organized as follows. In the next section we examine the regularity. Then in Section 3, we study the stability. In Section 4, we consider the stabilization. Finally, in Section 5 an example is given to show typical regularity of the saturation. As a general remark, all primary theorems are stated first and their proofs are then presented in a series of lemmas. We close this section with a few remarks. While we consider homogeneous right-hand sides in the first two equations of (1.1) here for simplicity, the later results can be extended to the nonhomogeneous case [10]. Also, all functions of s are assumed to be explicitly independent of x and t; otherwise, only lower-order terms appear in (1.4) and (1.5) and the subsequent analysis is the same. The theoretical results established in this paper are very useful in the choice and analysis of numerical methods for solving flow problems [5,6,11,12,14,15,19,25]. Since the differential system for the single-phase, miscible displacement of one incompressible fluid by another in porous media resembles that for the two-phase incompressible flow studied here [13,19], the analysis presented in this paper extends to the miscible displacement problem. Finally, we mention that the continuity of the saturation is known; see [1,16] under the assumptions that one of the degeneracies is at most logarithmic and is of power type, respectively. To prove the results in this paper, the continuity of the saturation is not enough; we need the Hölder continuity of this quantity.

2. Regularity of a weak solution

Define the spaces

$$V = \{v \in H^1(\Omega) : v|_{\Gamma_2^p} = 0;\ \text{if}\ \Gamma_2^p = \emptyset,\ \text{then} \int_\Omega v\,dx = 0\},$$
$$W = \{v \in H^1(\Omega) : v|_{\Gamma_2^\theta} = 0\}.$$

Below V^* and W^* indicate the duals of V and W, respectively. As mentioned in the introduction, existence of a weak solution to the system in (1.4) and (1.5) was established in [10] with

$$\theta \in L^2(J; W) + \varphi_4, \quad p \in L^\infty(J; V) + \varphi_2, \quad s = \mathscr{S}(\theta), \quad \phi\partial_t s \in L^2(J; W^*),$$

where the usual Sobolev spaces are used. Also, if the data are assumed to be physically consistent, it was shown via a maximum principle that $0 \leqslant s \leqslant 1$ a.e. on Ω_T [10], where $\Omega_T = \Omega \times J$. Thus, all functions of s need to be defined only on $[0, 1]$.

2.1. Main regularity results

Set

$$a(s) = -\frac{\lambda_w \lambda_0}{\lambda}\frac{\partial p_c}{\partial s}. \tag{2.1}$$

The assumptions described in this paper are required only for establishing the regularity, stability, and stabilization results. To see the assumptions needed for the existence result, see [10]. In this section, we need assumptions (A1)–(A10) below for regularity. First, we assume that

(A1) The porosity satisfies that $\phi^* \geqslant \phi(x) \geqslant \phi_* > 0$, and the permeability $\kappa(x)$ is a bounded, symmetric, and uniformly positive definite matrix; i.e.,

$$0 < \kappa_* \leqslant |\xi|^{-2} \sum_{i,j=1}^{d} \kappa_{ij}(x)\xi_i\xi_j \leqslant \kappa^* < \infty, \quad x \in \Omega, \ \xi \neq 0 \in \mathfrak{R}^d.$$

(A2) $\lambda(s)$, $a(s)$, and $\gamma_i(s)$ $(i = 1, 2, 3)$ are continuous in $s \in [0, 1]$. Furthermore, there are positive constants λ_*, λ^*, p_{c*}, and C^* such that

$$\lambda_* \leqslant \lambda(s) \leqslant \lambda^*, \quad -p_{c*} \leqslant \frac{\partial p_c}{\partial s}, \quad a(s) + |\gamma_1(s)| + |\gamma_2(s)| + |\gamma_3(s)| \leqslant C^*,$$
$$s \in [0, 1].$$

(A3) There are positive constants $\delta < 1/2$, α_i $(i = 1, \ldots, 4)$, and C_i $(i = 1, \ldots, 6)$ such that

$$\begin{aligned} &C_1 s^{\alpha_1} \leqslant a(s) \leqslant C_2 s^{\alpha_2}, && s \in [0, \delta], \\ &C_3 \leqslant a(s) \leqslant C_4, && s \in [\delta, 1-\delta], \\ &C_5(1-s)^{\alpha_3} \leqslant a(s) \leqslant C_6(1-s)^{\alpha_4}, && s \in [1-\delta, 1]. \end{aligned}$$

Note that assumption (A3) reflects the degeneracy of $a(s)$ near zero and one, and all three assumptions are physical reasonable. Below C (with or without a subscript) indicates a generic constant, which probably takes on different values in different occurrences.

Theorem 2.1 (Interior regularity of s). *Under assumptions* (A1)–(A3), *s is locally Hölder continuous in Ω_T. That is, for every compact set K of Ω_T, there exist constants $C > 0$ and $\alpha \in (0, 1)$ such that*

$$|s(x_1, t_1) - s(x_2, t_2)| \leqslant C(|x_1 - x_2|^{\alpha} + |t_1 - t_2|^{\alpha/2}),$$

for every pair of points (x_1, t_1), $(x_2, t_2) \in K$.

The constant C depends on the data and the distance from K to Γ, while $\alpha \leqslant \min\{\alpha_i, \ i = 1, \ldots, 4\}$ depends only on the data. We need an additional assumption for a corresponding result on u:

(A4) $\lambda(s)$ and $\gamma_1(s)$ are Hölder continuous in $s \in [0, 1]$.

Theorem 2.2 (Interior regularity of u). *Under assumptions* (A1)–(A4), *u is locally Hölder continuous in Ω. Namely, for every compact set K of Ω, there are constants $C > 0$, $\varepsilon_0 > 0$, and $\beta \in (0, 1)$ depending only on the data such that*

$$|u(x_1, t) - u(x_2, t)| \leqslant C|x_1 - x_2|^{\beta} \quad \forall t \geqslant \varepsilon_0,$$

for every pair of points $x_1, x_2 \in K$.

We now state global regularity results on s and u. Toward that end, we need assumptions on Γ and the boundary and initial data. For simplicity, in this paper we only consider the case where $\Gamma = \Gamma_1^p = \Gamma_1^\theta$ or $\Gamma = \Gamma_2^p = \Gamma_2^\theta$;

i.e., we consider either the complete Dirichlet case or the complete Newmann case. Under the assumption that Γ is sufficiently smooth (e.g., Γ is in the class H^2_* [3]), a solution of the present problem with the mixed boundary condition (1.8) can be continued to a neighborhood of $\bar{\Gamma}_1^p \cap \bar{\Gamma}_2^p$ or $\bar{\Gamma}_1^s \cap \bar{\Gamma}_2^s$, so the subsequent analysis reduces to such a case.

We first consider the Dirichlet boundary problem. For this, we need the assumptions:

(A5) $\Gamma_2^\theta = \Gamma$ satisfies the property of positive geometric density (see [23] for the definition).

(A6) s_0 is Hölder continuous on $\bar{\Omega}$.

(A7) φ_4 is Hölder continuous on $\Gamma \times J$.

Theorem 2.3 (Global regularity of s in the Dirichlet case). *Under assumptions* (A1)–(A3) *and* (A5)–(A7), *s is Hölder continuous on $\bar{\Omega}_T$. That is, there exist constants $C > 0$ and $\alpha \in (0, 1)$ such that*

$$|s(x_1, t_1) - s(x_2, t_2)| \leqslant C(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/2}),$$

for every pair of points (x_1, t_1), $(x_2, t_2) \in \bar{\Omega}_T$.

Again, the constants $C > 0$ and α depend only on the data. The exponent α in Theorem 2.3 may be different from that in Theorem 2.1. For notational convenience, we utilize the same exponent α. This remark applies to β, too. For a corresponding result on u, we require that

(A8) $\Gamma_2^p = \Gamma$ belongs to the class $C^{1+\beta}$ ($\beta \in (0, 1)$) and $\varphi_2 \in L^\infty(J; C^{1+\beta}(\bar{\Omega}))$.

Theorem 2.4 (Global regularity of u in the Dirichlet case). *Under assumptions* (A1)–(A4) *and* (A6)–(A8), *u is Hölder continuous on $\bar{\Omega}$ (uniformly in $t \in J$). Namely, there are constants $C > 0$ and $\beta \in (0, 1)$ such that*

$$|u(x_1, t) - u(x_2, t)| \leqslant C|x_1 - x_2|^\beta \quad \forall t \in J,$$

for every pair of points x_1, $x_2 \in \bar{\Omega}$. In particular, $u \in L^\infty(\Omega_T)$.

We now consider the Newmann boundary problem. In this case, we assume that

(A9) $\Gamma_1^\theta = \Gamma$ is of class $C^{1+\alpha}$ and $\varphi_3 \in L^2(J; W^{1,\infty}(\Omega))$.

Theorem 2.5 (Global regularity of s in the Newmann case). *Under assumptions* (A1)–(A3), (A6), *and* (A9), *s is Hölder continuous on $\bar{\Omega}_T$.*

Finally, we make the assumption in this section:

(A10) $\Gamma_1^p = \Gamma$ is of class $C^{1+\beta}$ and φ_1 is Hölder continuous on $\bar{\Omega}_T$.

Theorem 2.6 (Global regularity of u in the Newmann case). *Under assumptions* (A1)–(A4), (A6), *and* (A10), *u is Hölder continuous on $\bar{\Omega}$* (*uniformly in $t \in J$*). *In particular, $u \in L^\infty(\Omega_T)$.*

We shall only prove Theorem 2.1. Theorems 2.3 and 2.5 follow by combining the arguments in the proof of this theorem and those presented in [17] for handling boundary regularity. Under assumption (A4) and Theorem 2.1 (respectively, Theorems 2.3 and 2.5), the pressure p satisfies the elliptic equation (1.4) with Hölder continuous coefficients. Theorem 2.2 (respectively, Theorems 2.4 and 2.6) thus follows from the standard elliptic theory [20]. An example will be given in Section 5, which shows typical regularity of s. Specifically, in general s is only continuous or Hölder continuous, and its derivatives in space are discontinuous.

2.2. Preliminaries

In this subsection, we introduce notation which will be used in the later subsections. For any real number l, define the truncations of the saturation s by

$$(s-l)_+ = \max\{s-l, 0\}, \quad (s-l)_- = \max\{-(s-l), 0\}.$$

Also, for $\rho > 0$, define the cube

$$K_\rho = \left\{ x \in \Re^d : \max_{1 \leqslant i \leqslant d} |x_i| < \rho \right\}.$$

For $x_0 \in \Re^d$, let $x_0 + K_\rho$ denote the cube of center x_0. Also, for $\eta > 0$ a given number, define

$$Q(\eta, \rho) = K_\rho \times (-\eta, 0).$$

For $(x_0, t_0) \in \Re^{d+1}$, let $(x_0, t_0) + Q(\eta, \rho)$ be the "cylinder" congruent to $Q(\eta, \rho)$; i.e.,

$$(x_0, t_0) + Q(\eta, \rho) = \{x_0 + K_\rho\} \times (t_0 - \eta, t_0).$$

To obtain the Hölder continuity of a solution to degenerate parabolic problems, we need to work with cylinders whose dimensions are suitably scaled to reflect the degeneracy of the problems [17]. Let $\varepsilon > 0$ be a small number and consider the cylinder

$$(x_0, t_0) + Q(R^{2-\varepsilon}, 2R),$$

where $R > 0$ is so small that such a cylinder is completely contained in Ω_T. Set

$$s^+ = \operatorname{ess\,sup}\{s \,|\, (x_0, t_0) + Q(R^{2-\varepsilon}, 2R)\},$$
$$s^- = \operatorname{ess\,inf}\{s \,|\, (x_0, t_0) + Q(R^{2-\varepsilon}, 2R)\}, \quad \omega = s^+ - s^-.$$

For problem (1.5), these cylinders need to be suitably rescaled to reflect the two degeneracies of this problem.

Remark 2.1. For notational simplicity, in (A3) we assume that $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$, and define

$$\psi_0(s) = s^{\alpha_1}, \quad \psi_1(s) = s^{\alpha_3}.$$

From the later arguments, we shall see that this condition is not essential. Also, we shall present the subsequent analysis with the case $\alpha_1 = \alpha_3$. Otherwise, we shall work with the functions $\hat{\psi}_0(s)$ and $\hat{\psi}_1(s)$ defined by

$$\hat{\psi}_0(s) = \psi_0^{\alpha_3}(s), \quad \hat{\psi}_1(s) = \psi_1^{\alpha_1}(s).$$

Construct the cylinder

$$(x_0, t_0) + Q\left(\psi_1^{-1}\left(\frac{\omega}{2^m}\right)R^2, R\right),$$

where m is a positive integer. We assume that m is so large that we have the inclusion

$$Q\left(\psi_1^{-1}\left(\frac{\omega}{2^m}\right)R^2, R\right) \subset Q(R^{2-\varepsilon}, 2R). \tag{2.2}$$

Let m_0 be the smallest positive integer satisfying

$$\frac{\omega}{2^{m_0}} < \delta, \tag{2.3}$$

where δ is given in assumption (A3). Now, choose m large enough so that

$$\psi_1\left(\frac{\omega}{2^m}\right) \leqslant \frac{1}{2}\psi_0\left(\frac{\omega}{2^{m_0+2}}\right). \tag{2.4}$$

Inequality (2.4) relates the arguments in $\psi_0(\cdot)$ and $\psi_1(\cdot)$. Also, construct the cylinder

$$(x_0, \bar{t}) + Q\left(\psi_0^{-1}\left(\frac{\omega}{2^{m_0+2}}\right)R^2, R\right). \tag{2.5}$$

Under (2.4), if we have

$$\bar{t} - \psi_0^{-1}\left(\frac{\omega}{2^{m_0+2}}\right)R^2 > t_0 - \psi_1^{-1}\left(\frac{\omega}{2^m}\right)R^2, \tag{2.6}$$

then the following inclusion holds:

$$(x_0, \bar{t}) + Q\left(\psi_0^{-1}\left(\frac{\omega}{2^{m_0+2}}\right)R^2, R\right) \subset (x_0, t_0) + Q\left(\psi_1^{-1}\left(\frac{\omega}{2^m}\right)R^2, R\right). \tag{2.7}$$

We shall work with the subcylinders of the type in (2.5). For expositional convenience, we introduce the notation

$$\bar{Q}_R = Q\left(\psi_0^{-1}\left(\frac{\omega}{2^{m_0+2}}\right)R^2, R\right).$$

These subcylinders reflect the degeneracy at zero, and the degeneracy at one via (2.4). After a translation, we will work with $(x_0, t_0) = (0, 0)$ below.

Lemma 2.1 (Interior regularity of p). *Under assumptions* (A1) *and* (A2), *p is locally Hölder continuous in Ω (uniformly in t). Namely, for every compact set K of Ω, there are constants $C > 0$ and $\beta \in (0, 1)$ depending only on the data such that*

$$|p(x_1, t) - p(x_2, t)| \leqslant C|x_1 - x_2|^\beta \quad \forall t \in J,$$

for every pair of points $x_1, x_2 \in K$.

Proof. By (1.4), p satisfies the equation

$$-\nabla \cdot \{\kappa(\lambda(s)\nabla p + \gamma_1(s))\} = 0, \tag{2.8}$$

which is uniformly elliptic by assumptions (A1) and (A2). Then Lemma 2.1 follows from the standard elliptic theory [20]. $\square$

In the subsequent analysis, we shall fix such a compact set K of Ω from Lemma 2.1, where p is Hölder continuous with exponent β, and from now on we shall assume that $K_\rho \subset K$.

Below $(\cdot, \cdot)_S$ denotes the $L^2(S)$ inner product (or sometimes the duality pairing); S is omitted if $S = \Omega$. Also, ε_1 is a positive constant, as small as we please.

Lemma 2.2. *Under assumptions* (A1) *and* (A2), *for every $K_\rho \subset K$ there is a constant C depending only on the data such that*

$$(|\nabla p|^2, f^2)_{K_\rho} \leqslant C\{(1, f^2)_{K_\rho} + \rho^{2\beta}(1, |\nabla f|^2)_{K_\rho}\},$$

for all $f \in H_0^1(K_\rho)$, where β is from Lemma 2.1.

Proof. For any fixed $x_1 \in K_\rho$, multiply (2.8) by $(p(x, t) - p(x_1, t))f^2$, integrate the resulting equation over K_ρ, and use Green's formula to see that

$$\begin{aligned}&(\kappa(\lambda(s)\nabla p + \gamma_1(s)), f^2\nabla p)_{K_\rho} + 2(\kappa(\lambda(s)\nabla p + \gamma_1(s)),\\ &\quad (p(x, t) - p(x_1, t))f\nabla f)_{K_\rho} = 0.\end{aligned}$$

Then the desired result follows from the Hölder inequality and Lemma 2.1. $\square$

2.3. Proof of Theorem 2.1, Part I

The proof of Theorem 2.1 is carried out via an alternative argument introduced in [17]. Here special care must be taken on treating the two degeneracies of the coefficient $a(s)$ and the coupling of the saturation and pressure equations. To fix ideas and avoid repetition, we pay attention only to the arguments which are different from those in [17].

Lemma 2.3. *There is a constant $v_0 \in (0, 1)$, depending only on the data, such that if for some cylinder $(0, \bar{t}) + \bar{Q}_R$ it holds that*

$$\left|(x, t) \in \{(0, \bar{t}) + \bar{Q}_R\} \mid s(x, t) < s^- + \frac{\omega}{2^{m_0}}\right| \leqslant v_0|\bar{Q}_R|,$$

then we have either

$$\frac{\omega}{2^{m_0}} \leqslant CR^{\beta/2}, \tag{2.9}$$

or

$$s(x,t) > s^- + \frac{\omega}{2^{m_0+1}}, \qquad a.e.\ (x,t) \in (0,\bar{t}) + \bar{Q}_{R/2}, \tag{2.10}$$

where $|\cdot|$ *indicates the Lebesgue measure.*

Proof. Without loss of generality, we assume that $\bar{t} = 0$ and $s^- = 0$. Set

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \qquad n = 0, 1, 2, \dots .$$

We work with the cylinders $\bar{Q}_{R_n}$. Let $\xi_n(x,t)$ be a smooth cutoff function in $\bar{Q}_{R_n}$ satisfying

$$\begin{aligned}
&0 \leqslant \xi_n \leqslant 1 && \text{on } \bar{Q}_{R_n},\\
&\xi_n = 1 && \text{on } \bar{Q}_{R_{n+1}},\\
&\xi_n = 0 && \text{on } \partial\bar{Q}_{R_n} \text{ or for } t = -\psi_0^{-1}\left(\frac{\omega}{2^{m_0+2}}\right)R^2,\\
&|\nabla \xi_n| \leqslant 2^{n+1}/R, && |\Delta \xi_n| \leqslant C2^{2(n+1)}/R^2,\\
&0 \leqslant \partial_t \xi_n \leqslant 2^{2(n+1)}\psi_0\left(\frac{\omega}{2^{m_0+2}}\right)\Big/R^2.
\end{aligned}$$

Also, let

$$K_n = K_{R_n}, \quad s_\omega = \max\left\{s; \frac{\omega}{2^{m_0+2}}\right\}, \quad \hat{t} = -\psi_0^{-1}\left(\frac{\omega}{2^{m_0+2}}\right)R^2,$$

and

$$k_n = \frac{\omega}{2^{m_0+1}} + \frac{\omega}{2^{m_0+1+n}}, \quad n = 0, 1, 2, \dots .$$

Multiply (1.5) by $(s_\omega - k_n)_- \xi_n^2$, integrate over $K_n \times (\hat{t}, t)$ with $\hat{t} \leqslant t \leqslant 0$, and apply Green's formula to see that

$$\begin{aligned}
&\int_{\hat{t}}^{t} (\phi\partial_t s, (s_\omega - k_n)_- \xi_n^2)_{K_n}\, d\tau\\
&\quad + \int_{\hat{t}}^{t} (\kappa\{\nabla\theta + \gamma_2(s)\} + u\gamma_3(s), \nabla\{(s_\omega - k_n)_- \xi_n^2\})_{K_n}\, d\tau = 0.
\end{aligned} \tag{2.11}$$

We estimate each term in (2.11) as follows. First, note that

$$\begin{aligned}
&\int_{\hat{t}}^{t} (\phi\partial_t s, (s_\omega - k_n)_- \xi_n^2)_{K_n}\, d\tau\\
&\quad = \int_{\hat{t}}^{t} \Bigg\{ (\phi\partial_t (s_\omega - k_n)_-, (s_\omega - k_n)_- \xi_n^2)_{K_n}.\\
&\qquad + \left(\phi\partial_t\left(s - \frac{\omega}{2^{m_0+2}}\right)_-, \left(\frac{\omega}{2^{m_0+2}} - k_n\right)_- \xi_n^2\right)_{K_n} \Bigg\}\, d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(\phi(s_\omega - k_n)_-^2, \xi_n^2)_{K_n}(t) + \left(\phi\left(s - \frac{\omega}{2^{m_0+2}}\right)_-, \left(\frac{\omega}{2^{m_0+2}} - k_n\right)_- \xi_n^2\right)_{K_n}(t) \\
&\quad - \int_{\hat t}^t \Bigg\{ (\phi(s_\omega - k_n)_-^2, \xi_n \partial_t \xi_n)_{K_n}. \\
&\quad + 2(\phi(s - \frac{\omega}{2^{m_0+2}})_- \left(\frac{\omega}{2^{m_0+2}} - k_n\right)_-, \xi_n \partial_t \xi_n)_{K_n} \Bigg\} \, d\tau \\
&\geqslant \frac{1}{2}(\phi(s_\omega - k_n)_-^2, \xi_n^2)_{K_n}(t) - \int_{\hat t}^t (\phi(s_\omega - k_n)_-^2, \xi_n \partial_t \xi_n)_{K_n} \, d\tau \\
&\quad - C\left(\frac{\omega}{2^{m_0+2}}\right)^2 \int_{\hat t}^t (\chi\{s_\omega < k_n\}, \xi_n \partial_t \xi_n)_{K_n} \, d\tau,
\end{aligned}
$$

where $\chi\{s_\omega < k_n\}$ is the characteristic function of the set $\{s_\omega < k_n\}$. Thus, by the properties of ξ_n, we see that

$$
\begin{aligned}
\int_{\hat t}^t (\phi \partial_t s, (s_\omega - k_n)_- \xi_n^2)_{K_n} \, d\tau \geqslant \frac{1}{2}(\phi(s_\omega - k_n)_-^2, \xi_n^2)_{K_n}(t) - C\psi_0\left(\frac{\omega}{2^{m_0+2}}\right) \\
\left(\frac{\omega}{2^{m_0}}\right)^2 \left(\frac{2^{n+1}}{R}\right)^2 \int_{\hat t}^t |\{s_\omega < k_n\}| \, d\tau.
\end{aligned}
$$

Second, by (2.1), we have

$$
\begin{aligned}
&\int_{\hat t}^t (\kappa \nabla \theta, \nabla\{(s_\omega - k_n)_- \xi_n^2\})_{K_n} \, d\tau \\
&\quad = \int_{\hat t}^t (\kappa a(s) \nabla s, \nabla\{(s_\omega - k_n)_- \xi_n^2\})_{K_n} \, d\tau \\
&\quad = \int_{\hat t}^t \{(\kappa a(s_\omega)|\nabla(s_\omega - k_n)_-|^2, \xi_n^2)_{K_n} + 2(\kappa a(s) \nabla s, (s_\omega - k_n)_- \xi_n \nabla \xi_n)_{K_n}\} \, d\tau \\
&\quad \equiv H_1 + 2H_2.
\end{aligned}
$$

On the set $\{\frac{\omega}{2^{m_0+2}} < s_\omega < \frac{\omega}{2^{m_0}}\}$, by (A1), (A3), and (2.3), we see that

$$
H_1 \geqslant C\psi_0\left(\frac{\omega}{2^{m_0+2}}\right) \int_{\hat t}^t (|\nabla(s_\omega - k_n)_-|^2, \xi_n^2)_{K_n} \, d\tau.
$$

Also, by Green's formula, we observe that

$$
\begin{aligned}
H_2 &= \int_{\hat t}^t \Bigg\{ (\kappa a(s_\omega)(s_\omega - k_n)_- \nabla(s_\omega - k_n)_-, \xi_n \nabla \xi_n)_{K_n} \\
&\quad + \left(\kappa a(s)\left(\frac{\omega}{2^{m_0+2}} - k_n\right)_- \nabla\left(s - \frac{\omega}{2^{m_0+2}}\right)_-, \xi_n \nabla \xi_n\right)_{K_n} \Bigg\} \, d\tau \\
&= \int_{\hat t}^t \Bigg\{ (\kappa a(s_\omega)(s_\omega - k_n)_- \nabla(s_\omega - k_n)_-, \xi_n \nabla \xi_n)_{K_n} \\
&\quad + \left(\kappa \nabla\left(\int_{\frac{\omega}{2^{m_0+2}}}^s a(\xi) \, d\xi\right)_- \left(\frac{\omega}{2^{m_0+2}} - k_n\right)_-, \xi_n \nabla \xi_n\right)_{K_n} \Bigg\} \, d\tau
\end{aligned}
$$

$$
\begin{aligned}
= \int_{\hat{t}}^{t} \Bigg\{ & (\kappa a(s_\omega)(s_\omega - k_n)_- \nabla (s_\omega - k_n)_-, \xi_n \nabla \xi_n)_{K_n} \\
& - \left(\nabla \cdot \kappa \left(\int_{\frac{\omega}{2^{m_0+2}}}^{s} a(\xi)\, d\xi \right)_- \left(\frac{\omega}{2^{m_0+2}} - k_n \right)_-, \xi_n \nabla \xi_n \right)_{K_n} \\
& - \left(\kappa \left(\int_{\frac{\omega}{2^{m_0+2}}}^{s} a(\xi)\, d\xi \right)_- \left(\frac{\omega}{2^{m_0+2}} - k_n \right)_-, |\nabla \xi_n|^2 \right)_{K_n} \\
& - \left(\kappa \left(\int_{\frac{\omega}{2^{m_0+2}}}^{s} a(\xi)\, d\xi \right)_- \left(\frac{\omega}{2^{m_0+2}} - k_n \right)_-, \xi_n \Delta \xi_n \right)_{K_n} \Bigg\} d\tau,
\end{aligned}
$$

so, by the properties of ξ_n again,

$$
\begin{aligned}
|H_2| \leqslant \psi_0\left(\frac{\omega}{2^{m_0+2}}\right) \int_{\hat{t}}^{t} \Bigg\{ & \varepsilon_1 (|\nabla (s_\omega - k_n)_-|^2, \xi_n^2)_{K_n} \\
& + C\left(\frac{\omega}{2^{m_0}}\right)^2 \left(\frac{2^{n+1}}{R}\right)^2 |\chi\{s_\omega < k_n\}| \Bigg\} d\tau.
\end{aligned}
$$

Third, by (2.1) and (A2), we have

$$
|\gamma_2(s)| \leqslant Ca(s). \tag{2.12}
$$

Then, we see that

$$
\begin{aligned}
& \int_{\hat{t}}^{t} (\kappa \gamma_2(s), \nabla \{(s_\omega - k_n)_- \xi_n^2\})_{K_n}\, d\tau \\
& \quad = \int_{\hat{t}}^{t} \{ (\kappa \gamma_2(s) \nabla (s_\omega - k_n)_-, \xi_n^2)_{K_n} + 2(\kappa \gamma_2(s)(s_\omega - k_n)_-, \xi_n \nabla \xi_n)_{K_n} \}\, d\tau \\
& \quad \leqslant \psi_0\left(\frac{\omega}{2^{m_0+2}}\right) \int_{\hat{t}}^{t} \Bigg\{ \varepsilon_1 (|\nabla (s_\omega - k_n)_-|^2, \xi_n^2)_{K_n} \\
& \qquad + C\left(1 + \frac{\omega}{2^{m_0}} \frac{2^{n+1}}{R}\right) |\chi\{s_\omega < k_n\}| \Bigg\} d\tau.
\end{aligned}
$$

Fourth, by the choice of m_0 in (2.3) and the definition of γ_3, note that an inequality similar to (2.12) also holds for γ_3. Then it follows from (1.4) and Green's formula that

$$
\begin{aligned}
& \int_{\hat{t}}^{t} (u \gamma_3(s), \nabla \{(s_\omega - k_n)_- \xi_n^2\})_{K_n}\, d\tau \\
& \quad = \int_{\hat{t}}^{t} \{ (u \gamma_3(s) \nabla (s_\omega - k_n)_-, \xi_n^2)_{K_n} + 2(u \gamma_3(s)(s_\omega - k_n)_-, \xi_n \nabla \xi_n)_{K_n} \}\, d\tau
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\hat{t}}^{t} \left\{ \left(u\nabla \left(\int_{k_n}^{s_\omega} \gamma_3(\xi)\, d\xi \right)_{-}, \xi_n^2 \right)_{K_n} \right. \\
&\quad \left. + 2(u\gamma_3(s)(s_\omega - k_n)_-, \xi_n \nabla \xi_n)_{K_n} \right\} d\tau \\
&\leqslant C\psi_0\left(\frac{\omega}{2^{m_0+2}}\right) \int_{\hat{t}}^{t} \{(|\nabla p|(s_\omega - k_n)_-, \xi_n |\nabla \xi_n|)_{K_n} \\
&\quad + ((s_\omega - k_n)_-, \xi_n |\nabla \xi_n|)_{K_n}\}\, d\tau \\
&\leqslant C\psi_0\left(\frac{\omega}{2^{m_0+2}}\right) \int_{\hat{t}}^{t} \{\varepsilon_1 R_n^{-\beta}(|\nabla p|^2, (s_\omega - k_n)_-^2 \xi_n^2)_{K_n} \\
&\quad + R_n^{\beta}(|\nabla \xi_n|^2, \chi\{s_\omega < k_n\})_{K_n} \\
&\quad + ((s_\omega - k_n)_-, \xi_n |\nabla \xi_n|)_{K_n}\}\, d\tau.
\end{aligned}
$$

Hence, by Lemma 2.2 with $f = (s_\omega - k_n)_- \xi_n$, we see that

$$
\begin{aligned}
&\int_{\hat{t}}^{t} (u\gamma_3(s), \nabla\{(s_\omega - k_n)_- \xi_n^2\})_{K_n}\, d\tau \\
&\leqslant C\psi_0\left(\frac{\omega}{2^{m_0+2}}\right) \int_{\hat{t}}^{t} \left\{ \varepsilon_1 R_n^{\beta}(|\nabla (s_\omega - k_n)_-|^2, \xi_n^2)_{K_n} \right. \\
&\quad \left. + \left[R_n^{-\beta}\left(\frac{\omega}{2^{m_0}}\right)^2 + R_n^{\beta}\left(\frac{2^{n+1}}{R}\right)^2 + \frac{\omega}{2^{m_0}} \frac{2^{n+1}}{R} \right] |\chi\{s_\omega < k_n\}| \right\} d\tau.
\end{aligned}
$$

If (2.9) is violated, we have

$$
CR^{\beta/2} < \frac{\omega}{2^{m_0}}.
$$

Substitute all these inequalities into (2.11) and choose ε_1 appropriately to obtain

$$
\begin{aligned}
&((s_\omega - k_n)_-^2, \xi_n^2)_{K_n}(t) + \psi_0\left(\frac{\omega}{2^{m_0+2}}\right) \int_{\hat{t}}^{t} (|\nabla (s_\omega - k_n)_-|^2, \xi_n^2)_{K_n}\, d\tau \\
&\leqslant C\psi_0\left(\frac{\omega}{2^{m_0+2}}\right) \left(\frac{\omega}{2^{m_0}}\right)^2 \left(\frac{2^{n+1}}{R}\right)^2 \int_{\hat{t}}^{t} |\chi\{s_\omega < k_n\}|\, d\tau. \qquad (2.13)
\end{aligned}
$$

Introduce the change of time variable $z = t\psi_0(\frac{\omega}{2^{m_0+2}})$, which transfers

$$
Q\left(\psi_0^{-1}\left(\frac{\omega}{2^{m_0+2}}\right) R_n^2, R_n\right) \quad \text{into} \quad Q(R_n^2, R_n).
$$

Set

$$
v(\cdot, z) = s_\omega\left(\cdot, t\psi_0^{-1}\left(\frac{\omega}{2^{m_0+2}}\right)\right), \quad \hat{\xi}_n(\cdot, z) = \xi_n\left(\cdot, t\psi_0^{-1}\left(\frac{\omega}{2^{m_0+2}}\right)\right).
$$

Then it follows from (2.13) that

$$\sup_{-R_n^2 \leqslant z \leqslant 0} ((v-k_n)_-^2, \hat{\xi}_n^2)_{K_n}(z) + \int_{-R_n^2}^0 (|\nabla(v-k_n)_-|^2, \hat{\xi}_n^2)_{K_n}\, dz$$
$$\leqslant C\left(\frac{\omega}{2^{m_0}}\right)^2 \left(\frac{2^{n+1}}{R}\right)^2 \int_{-R_n^2}^0 |\chi\{v < k_n\}|\, dz.$$

Now, the rest of the proof is completed by a standard argument for parabolic problems (see Lemma 4.1 in Chapter III of [17]). $\square$

We now suppose that the assumption of Lemma 2.3 is satisfied for some cylinder $(0, \bar{t}) + \bar{Q}_R$. Then, at time level

$$-\hat{t} = \bar{t} - \psi_0^{-1}\left(\frac{\omega}{2^{m_0+2}}\right)\left(\frac{R}{2}\right)^2,$$

the function $x \mapsto s(x, -\hat{t})$ is larger than $s^- + \frac{\omega}{2^{m_0+1}}$ in $K_{R/2}$ by Lemma 2.3. We consider the cylinder

$$Q(\hat{t}, R/2) = K_{R/2} \times (-\hat{t}, 0).$$

Set

$$H_\omega^- = \operatorname{ess\,sup}\left\{\left(s - (s^- + \frac{\omega}{2^{m_0+1}})\right)_- \mid Q(\hat{t}, R/2)\right\} \leqslant \frac{\omega}{2^{m_0+1}},$$

and define

$$\Psi = \ln^+\left\{\frac{H_\omega^-}{H_\omega^- - (s - (s^- + \frac{\omega}{2^{m_0+1}}))_- + \frac{\omega}{2^{m_0+1+n}}}\right\},$$

where $\ln^+ v = \max\{\ln v, 0\}$ and $n \geqslant 1$ is to be determined later.

The following result says that, because of (2.10), the set where $s(\cdot, t)$ is close to s^- within a smaller cube can be made arbitrarily small for all $t \in (-\hat{t}, 0)$.

Lemma 2.4. *For every $v_1(\omega) \in (0, 1)$, there is a positive integer $m_1 > m_0 + 1$ (independent of ω) such that either*

$$\frac{\omega}{2^{m_1}} \leqslant CR^{\beta/2} \tag{2.14}$$

or

$$\left| x \in K_{R/4} \mid s(x, t) < s^- + \frac{\omega}{2^{m_1}} \right| \leqslant v_1 |K_{R/4}|,$$

for all $t \in (-\hat{t}, 0)$.

Proof. Again, without loss of generality, let $s^- = 0$. Let $x \mapsto \zeta(x)$ be a smooth cutoff function in $K_{R/2}$ such that

$$\zeta(x) = 1 \text{ on } K_{R/4} \text{ and } \nabla\zeta| \leqslant CR^{-1} \text{ on } K_{R/2}.$$

Multiply (1.5) by $(\Psi^2(s))'\zeta^2$, integrate on $K_{R/2}\times(-\hat{t},t)$ $(t\leqslant 0)$, and apply Green's formula to see that

$$\int_{-\hat{t}}^{t}(\phi\partial_t s,(\Psi^2)'\zeta^2)_{K_{R/2}}\,d\tau + \int_{-\hat{t}}^{t}(\kappa\{\nabla\theta+\gamma_2(s)\}+u\gamma_3(s),\nabla\{(\Psi^2)'\zeta^2\})_{K_{R/2}}\,d\tau=0. \tag{2.15}$$

As in the proof of Lemma 2.3, we need to estimate each term in (2.15). First, for $t=-\hat{t}$, $s>s^-+\frac{\omega}{2^{m_0+1}}$, so $\Psi(x,-\hat{t})=0$ on $K_{R/2}$. Hence, we see that

$$\begin{aligned}&\int_{-\hat{t}}^{t}(\phi\partial_t s,(\Psi^2)'\zeta^2)_{K_{R/2}}\,d\tau\\&\quad=(\phi\Psi^2,\zeta^2)_{K_{R/2}}(t)-(\phi\Psi^2,\zeta^2)_{K_{R/2}}(-\hat{t})\geqslant(\phi,\Psi^2)_{K_{R/4}}(t).\end{aligned}$$

Second, by the property of ζ, we have

$$\begin{aligned}&\int_{-\hat{t}}^{t}(\kappa a(s)\nabla s,\nabla\{(\Psi^2)'\zeta^2\})_{K_{R/2}}\,d\tau\\&\quad=2\int_{-\hat{t}}^{t}\{(\kappa a(s)|\nabla s|^2,(1+\Psi)(\Psi')^2\zeta^2)_{K_{R/2}}+2(\kappa a(s)\nabla s,\Psi\Psi'\zeta\nabla\zeta)_{K_{R/2}}\}\,d\tau\\&\quad\geqslant C\int_{-\hat{t}}^{t}\{(a(s)|\nabla s|^2,(1+\Psi)(\Psi')^2\zeta^2)_{K_{R/2}}-R^{-2}(a(s),\Psi)_{K_{R/2}}\}\,d\tau.\end{aligned}$$

Third, we see that

$$\begin{aligned}&\int_{-\hat{t}}^{t}(\kappa\gamma_2(s),\nabla\{(\Psi^2)'\zeta^2\})_{K_{R/2}}\,d\tau\\&\quad=2\int_{-\hat{t}}^{t}\{(\kappa\gamma_2(s)\cdot\nabla s,(1+\Psi)(\Psi')^2\zeta^2)_{K_{R/2}}+2(\kappa\gamma_2(s),\Psi\Psi'\zeta\nabla\zeta)_{K_{R/2}}\}\,d\tau.\end{aligned}$$

Note that the above integrals extend only on the set $\{s<\frac{\omega}{2^{m_0+1}}\}\cap Q(\hat{t},R/2)$. Thus, by (2.12), we find that

$$\begin{aligned}&\int_{-\hat{t}}^{t}(\kappa\gamma_2(s),\nabla\{(\Psi^2)'\zeta^2\})_{K_{R/2}}\,d\tau\\&\quad\leqslant\int_{-\hat{t}}^{t}\{\varepsilon_1(a(s)|\nabla s|^2,(1+\Psi)(\Psi')^2\zeta^2)_{K_{R/2}}+C[(a(s)(1+\Psi),(\Psi')^2\zeta^2)_{K_{R/2}}\\&\qquad+(a(s)\Psi|\Psi'|,\zeta|\nabla\zeta|)_{K_{R/2}}]\}\,d\tau.\end{aligned}$$

By the definition of Ψ, we see that

$$\Psi\leqslant n\ln 2,\quad |\Psi'|\leqslant C\frac{2^{m_0+1+n}}{\omega}.$$

Hence we obtain

$$\begin{aligned}&\int_{-\hat{t}}^{t}(\kappa\gamma_2(s),\nabla\{(\Psi^2)'\zeta^2\})_{K_{R/2}}\,d\tau\\&\quad\leqslant\int_{-\hat{t}}^{t}\varepsilon_1(a(s)|\nabla s|^2,(1+\Psi)(\Psi')^2\zeta^2)_{K_{R/2}}\,d\tau\\&\quad\quad+\;C\psi_0\Big(\frac{\omega}{2^{m_0}}\Big)\frac{n}{R}\Big(\frac{2^{m_0+1+n}}{\omega}\Big)^2|Q(\hat{t},R/2)|.\end{aligned}$$

Fourth, using Lemma 2.2 with $f=\zeta$, a similar argument yields

$$\begin{aligned}&\int_{-\hat{t}}^{t}(u\gamma_3(s),\nabla\{(\Psi^2)'\zeta^2\})_{K_{R/2}}\,d\tau\\&\quad=\;2\int_{-\hat{t}}^{t}\{(u\gamma_3(s)\cdot\nabla s,(1+\Psi)(\Psi')^2\zeta^2)_{K_{R/2}}+2(u\gamma_3(s),\Psi\Psi'\zeta\nabla\zeta)_{K_{R/2}}\}\,d\tau\\&\quad\leqslant\int_{-\hat{t}}^{t}\{\varepsilon_1(a(s)|\nabla s|^2,(1+\Psi)(\Psi')^2\zeta^2)_{K_{R/2}}+C[(a(s)|u|^2(1+\Psi),(\Psi')^2\zeta^2)_{K_{R/2}}\\&\quad\quad+\;(|u|a(s)\Psi|\Psi'|,\zeta|\nabla\zeta|)_{K_{R/2}}]\}\,d\tau\\&\quad\leqslant\int_{-\hat{t}}^{t}\varepsilon_1(a(s)|\nabla s|^2,(1+\Psi)(\Psi')^2\zeta^2)_{K_{R/2}}\,d\tau\\&\quad\quad+\;C\psi_0(\frac{\omega}{2^{m_0}})n\left\{\Big(\frac{2^{m_0+1+n}}{\omega}\Big)^2\left[1+\frac{R^{2\beta}}{R^2}\right]+R^{-2}\right\}|Q(\hat{t},R/2)|.\end{aligned}$$

Substitute all these inequalities into (2.15) and choose ε_1 appropriately to find

$$(\Psi^2(t),1)_{K_{R/4}}\leqslant C\psi_0\Big(\frac{\omega}{2^{m_0}}\Big)\left(\frac{n}{R^2}+\Big(\frac{2^{m_0+1+n}}{\omega}\Big)^2\left[\frac{n}{R}+\frac{nR^{2\beta}}{R^2}\right]\right)|Q(\hat{t},R/2)|.$$

Notice that

$$|Q(\hat{t},R/2)|\leqslant CR^{d+2}/\psi_1\Big(\frac{\omega}{2^m}\Big),$$

where we recall that d is the dimension number of Ω and m is defined as in (2.2). Then it follows that

$$(\Psi^2(t),1)_{K_{R/4}}\leqslant CR^{d+2}\frac{\psi_0(\omega/2^{m_0})}{\psi_1(\omega/2^m)}\left\{\frac{n}{R^2}+\Big(\frac{2^{m_0+1+n}}{\omega}\Big)^2\left[\frac{n}{R}+\frac{nR^{2\beta}}{R^2}\right]\right\}. \tag{2.16}$$

If (2.14) is violated, apply (2.16) to see that

$$(\Psi^2(t),1)_{K_{R/4}}\leqslant CnR^d\frac{\psi_0(\omega/2^{m_0})}{\psi_1(\omega/2^m)}. \tag{2.17}$$

On the set

$$\left\{x\in K_{R/4} \mid s(x,t)<s^- + \frac{\omega}{2^{m_0+1+n}}\right\}, \quad t\in(-\hat{t},0),$$

the definition of Ψ means that

$$\Psi^2 \geqslant (n-1)^2 \ln^2 2.$$

Hence it follows from (2.17) that

$$\left|\left\{x\in K_{R/4} \mid s(x,t)<s^- + \frac{\omega}{2^{m_0+1+n}}\right\}\right| \\ \leqslant C\frac{n}{(n-1)^2}\frac{\psi_0(\omega/2^{m_0})}{\psi_1(\omega/2^m)}|K_{R/4}|, \quad t\in(-\hat{t},0),$$

which implies the desired result by choosing n large enough. $\square$

Remark 2.2. Because of Remark 2.1 (particularly because $\alpha_1 = \alpha_3$), m_1 in this lemma is independent of ω. If $\alpha_1 \neq \alpha_3$, the proof in this lemma still works if we work with the functions $\hat{\psi}_0(s)$ and $\hat{\psi}_1(s)$ in place of $\psi_0(s)$ and $\psi_1(s)$, respectively, as mentioned before. This remark applies to other positive integers m_i $(i=2,\dots,5)$ later.

Lemma 2.4 is now used to show that s is strictly bounded away from s^- in a smaller cylinder, as stated below.

Lemma 2.5. *Suppose that the assumption of Lemma* 2.3 *is satisfied for some cylinder* $(0,\bar{t})+\bar{Q}_R$. *Then there is a positive integer* $m_2>m_1$ (*independent of* ω) *such that either*

$$\frac{\omega}{2^{m_2}} \leqslant CR^{\beta/2}$$

or

$$s(x,t)>s^- + \frac{\omega}{2^{m_2+1}}, \quad \text{a.e. } (x,t)\in K_{R/8}\times(-\hat{t},0).$$

The proof of this lemma can be completed by combining the techniques in Lemma 2.3 and the argument in Lemma 6.2 in Chapter III of [17]. The results obtained so far are summarized in the next proposition; i.e., the so-called first alternative in [17] is established.

Proposition 2.1. *There is a constant* $v_0\in(0,1)$, *depending only on the data, and a positive integer* $m_2\geqslant 1$ (*independent of* ω) *such that if for some cylinder* $(0,\bar{t})+\bar{Q}_R$ *it holds that*

$$\left|(x,t)\in\{(0,\bar{t})+\bar{Q}_R\} \mid s(x,t)<s^- + \frac{\omega}{2^{m_0}}\right| \leqslant v_0|\bar{Q}_R|,$$

then we have either

$$\omega \leqslant C 2^{m_2} R^{\beta/2} \tag{2.18}$$

or

$$\operatorname{ess\,osc}\{s(x,t) \mid (x,t) \in \bar{Q}_{R/8}\} \leqslant \left(1 - \frac{1}{2^{m_2+1}}\right)\omega. \tag{2.19}$$

Proof. If (2.18) is violated, Lemma 2.5 implies that

$$\operatorname{ess\,inf}\{s \mid (x,t) \in \bar{Q}_{R/8}\} \geqslant s^- + \frac{\omega}{2^{m_2+1}}.$$

Hence,

$$\begin{aligned} &\operatorname{ess\,sup}\{s \mid (x,t) \in \bar{Q}_{R/8}\} - \operatorname{ess\,inf}\{s \mid (x,t) \in \bar{Q}_{R/8}\} \\ &\quad \leqslant \operatorname{ess\,sup}\{s \mid (x,t) \in \bar{Q}_{R/8}\} - s^- - \frac{\omega}{2^{m_2+1}} \\ &\quad \leqslant \left(1 - \frac{1}{2^{m_2+1}}\right)\omega, \end{aligned}$$

which is the desired result. □

2.4. Proof of Theorem 2.1, Part II

We now analyze the case where the assumption of Lemma 2.3 is violated; i.e., for every cylinder $(0,\bar{t}) + \bar{Q}_R$,

$$\left|(x,t) \in \{(0,\bar{t}) + \bar{Q}_R\} \mid s(x,t) < s^- + \frac{\omega}{2^{m_0}}\right| > v_0 |\bar{Q}_R|.$$

Note that

$$s^+ - \frac{\omega}{2^{m_0}} \geqslant s^- + \frac{\omega}{2^{m_0}} \qquad \text{for } m_0 \geqslant 2.$$

We can rewrite the above inequality as follows:

$$\left|(x,t) \in \{(0,\bar{t}) + \bar{Q}_R\} \mid s(x,t) > s^+ - \frac{\omega}{2^{m_0}}\right| \leqslant (1 - v_0)|\bar{Q}_R|, \tag{2.20}$$

which holds for all cylinders $(0,\bar{t}) + \bar{Q}_R$. In terms of (2.20), we examine the behavior of s near s^+. Let us fix one of such cylinders with "vertex" $(0,\bar{t})$ where

$$\bar{t} \in \left[-\frac{R^2}{\psi_1(\frac{\omega}{2^m})} + \frac{R^2}{\psi_0(\frac{\omega}{2^{m_0+2}})}, 0\right].$$

Lemma 2.6. *Under* (2.20), *there is a time level t^* in the interval*

$$\left(\bar{t} - \frac{R^2}{\psi_0(\frac{\omega}{2^{m_0+2}})}, \bar{t} - \frac{v_0}{2}\frac{R^2}{\psi_0(\frac{\omega}{2^{m_0+2}})}\right),$$

such that

$$\left| x \in K_R \mid s(x, t^*) > s^+ - \frac{\omega}{2^{m_0}} \right| \leqslant \frac{1 - v_0}{1 - v_0/2} |K_R|.$$

This statement is a simple consequence of (2.20) (see Lemma 7.1 in Chapter III of [17]). This lemma implies that at some time level t^* the set where s is close to s^+ occupies only part of the cube K_R. The next result asserts that this does occur for all time levels in a small interval.

Lemma 2.7. *Under* (2.20), *there is a positive integer* $m_3 > m_0$ (*independent of* ω) *such that either*

$$\frac{\omega}{2^{m_3}} \leqslant C R^{\beta/2}$$

or

$$\left| x \in K_R \mid s(x, t) > s^+ - \frac{\omega}{2^{m_3}} \right| \leqslant \left(1 - \frac{v_0^2}{4} \right) |K_R| \quad \forall t \in \left[\bar{t} - v_0 R^2 \psi_0^{-1} \left(\frac{\omega}{2^{m_0+2}} \right) \Big/ 2, \bar{t} \right]. \tag{2.21}$$

Proof. The proof is similar to that in Lemma 2.4. Set

$$H_\omega^+ = \operatorname{ess\,sup} \left\{ \left(s - \left(s^+ - \frac{\omega}{2^{m_0}} \right) \right)_+ \mid (0, \bar{t}) + \bar{Q}_R \right\} \leqslant \frac{\omega}{2^{m_0}}$$

and

$$\Psi = \ln^+ \left\{ \frac{H_\omega^+}{H_\omega^+ - (s - (s^+ - \frac{\omega}{2^{m_0}}))_+ + \frac{\omega}{2^{m_0+n}}} \right\},$$

where n is to be determined below. The cutoff function $x \mapsto \zeta(x)$ satisfies

$$\zeta(x) = 1 \text{ for } x \in K_{(1-\sigma)R} \quad \text{and} \quad |\nabla \zeta| \leqslant \frac{1}{\sigma R} \text{ on } K_R,$$

where $\sigma \in (0, 1)$. Multiply (1.5) by $(\Psi^2)' \zeta^2$, and utilize the same argument as in Lemma 2.4 to see that

$$(\Psi^2(t), 1)_{K_{(1-\sigma)R}} \leqslant (\Psi^2(t^*), 1)_{K_R} + \frac{Cn}{\sigma} \frac{\psi_1(\frac{\omega}{2^{m_0}})}{\psi_0(\frac{\omega}{2^{m_0}})} |K_R|.$$

Note that $\Psi \leqslant n \ln 2$ and it vanishes on the set $\{s < s^+ - \frac{\omega}{2^{m_0}}\}$. Hence, by Lemma 2.6, we have

$$(\Psi^2(t), 1)_{K_{(1-\sigma)R}} \leqslant Cn \left\{ \frac{n(1 - v_0)}{1 - v_0/2} + \frac{\psi_1(\frac{\omega}{2^{m_0}})}{\sigma \psi_0(\frac{\omega}{2^{m_0}})} \right\} |K_R|. \tag{2.22}$$

On the set $\{x \in K_{(1-\sigma)R} \mid s > s^+ - \frac{\omega}{2^{m_0+n}}\}$,

$$\Psi^2 \geqslant (n - 1)^2 \ln^2 2,$$

so it follows from (2.22) that

$$\left| x \in K_{(1-\sigma)R} \mid s > s^{+} - \frac{\omega}{2^{m_0+n}} \right| \leqslant \frac{Cn}{(n-1)^2} \left\{ \frac{n(1-\nu_0)}{1-\nu_0/2} + \frac{\psi_1(\frac{\omega}{2^{m_0}})}{\sigma \psi_0(\frac{\omega}{2^{m_0}})} \right\} |K_R|. \tag{2.23}$$

Also, observe that

$$\left| x \in K_R \mid s > s^{+} - \frac{\omega}{2^{m_0+n}} \right| \leqslant \left| x \in K_{(1-\sigma)R} \mid s > s^{+} - \frac{\omega}{2^{m_0+n}} \right| + \left| K_R \backslash K_{(1-\sigma)R} \right| \leqslant \left| x \in K_{(1-\sigma)R} \mid s > s^{+} - \frac{\omega}{2^{m_0+n}} \right| + d\sigma |K_R|.$$

Consequently, by (2.23), we see that

$$\left| x \in K_R \mid s > s^{+} - \frac{\omega}{2^{m_0+n}} \right| \leqslant C \left(\left(\frac{n}{n-1} \right)^2 \frac{1-\nu_0}{1-\nu_0/2} + \frac{1}{\sigma n} \frac{\psi_1(\frac{\omega}{2^{m_0}})}{\psi_0(\frac{\omega}{2^{m_0}})} + d\sigma \right) |K_R|.$$

Finally, choosing σ and n appropriately generates the desired result. $\square$

Since (2.20) is valid for all cylinders of the form $(0, \bar{t}) + \bar{Q}_R$, using (2.6), (2.21) holds for all t in the interval

$$\left(-\frac{R^2}{\psi_1(\frac{\omega}{2^m})} + \left(1 - \frac{\nu_0}{2} \right) \frac{R^2}{\psi_0(\frac{\omega}{2^{m_0+2}})}, 0 \right).$$

Then, using (2.4), we see that (2.21) holds for all $t \in (-\frac{R^2}{2\psi_1(\frac{\omega}{2^m})}, 0)$. Therefore, we work with cylinders of the type

$$Q_R^m = K_R \times \left(-\frac{R^2}{2\psi_1\left(\frac{\omega}{2^m}\right)}, 0 \right).$$

Lemma 2.8. *For every $\nu_2 \in (0,1)$ there is a positive integer $m_4 > m_3$ (independent of ω) such that either*

$$\frac{\omega}{2^{m_4}} \leqslant CR^{\beta/2}$$

or

$$\left| (x,t) \in Q_R^m \mid s(x,t) > s^{+} - \frac{\omega}{2^{m_4}} \right| \leqslant \nu_2 |Q_R^m|.$$

The proof of this lemma can be carried out by using the ideas of last subsection and the argument in Lemma 8.1 in Chapter III of [17]. The same remark applies to the following lemma (see Lemma 9.1 in Chapter III of [17]).

Lemma 2.9. *There are constants* $\sigma_* \in (0,1)$ *and* $m_5 > m_4$ (*independent of* ω) *such that either*

$$\frac{\omega}{2^{m_5}} \leqslant C R^{\beta/2}$$

or

$$s(x,t) < s^+ - \frac{\omega}{2^{m_5+1}}, \quad \text{a.e. } (x,t) \in K_{R/8} \times \left(-\frac{\sigma_*}{\psi_1(\frac{\omega}{2^m})} \left(\frac{R}{8}\right)^2, 0 \right).$$

Now we summarize the results of this subsection in the next proposition; i.e., the second alternative is shown.

Proposition 2.2. *Suppose that for all cylinders of the type* $(0,\bar{t}) + \bar{Q}_R$ *it holds that*

$$\left| (x,t) \in \{(0,\bar{t}) + \bar{Q}_R\} \mid s(x,t) > s^+ - \frac{\omega}{2^{m_0}} \right| \leqslant (1 - v_0)|\bar{Q}_R|.$$

Then there are constants $\sigma_* \in (0,1)$ *and* m_5 (*independent of* ω) *such that either*

$$\omega \leqslant C 2^{m_5} R^{\beta/2}$$

or

$$\text{ess osc}\left\{ s(x,t) \mid (x,t) \in K_{R/8} \times \left(-\frac{\sigma_*}{\psi_1(\frac{\omega}{2^m})} \left(\frac{R}{8}\right)^2, 0 \right) \right\} \leqslant \left(1 - \frac{1}{2^{m_5+1}}\right)\omega.$$

This proposition can be shown as in Proposition 2.1 by using Lemma 2.9. The two alternatives in Propositions 2.1 and 2.2 can be combined to prove Theorem 2.1 with a standard fashion (see Proposition 3.1 in Chapter III of [17]).

3. Stability of the weak solution

In this section, we prove stability results for the weak solution of the previous section with respect to the boundary and initial data. Uniqueness of this solution then follows trivially from this result. The stability results heavily depend on the results established in the last section; especially, those on the uniform boundedness of u in Theorems 2.4 and 2.6 are used.

3.1. Main stability results

For the next two results we need the assumption below. Its meaning will be described in Section 3.3.

(A11) There is a constant $C>0$ such that

$$\|\lambda(s_1)-\lambda(s_2)\|^2_{L^2(\Omega)}+\sum_{i=1}^{3}\|\gamma_i(s_1)-\gamma_i(s_2)\|^2_{L^2(\Omega)}\leqslant C(s_1-s_2,\theta_1-\theta_2),$$
$$0\leqslant\theta_1,\ \theta_2\leqslant\theta^*,\ \ s_i=\mathscr{S}(\theta_i),\ i=1,2.$$

Let (s_1,θ_1,p_1,u_1) and (s_2,θ_2,p_2,u_2) solve the system in (1.4) and (1.5) with the boundary and initial data $(\varphi_1^1,\varphi_2^1,\varphi_3^1,\varphi_4^1,s_0^1)$ and $(\varphi_1^2,\varphi_2^2,\varphi_3^2,\varphi_4^2,s_0^2)$, respectively. Again, we first consider the Dirichlet boundary problem.

Theorem 3.1 (Stability in the Dirichlet case). *In addition to the assumptions of Theorem* 2.4, *if* (A11) *is satisfied*, *then*

$$\begin{aligned}&\|s_1-s_2\|^2_{L^\infty(J;H^{-1}(\Omega))}+\int_J(s_1-s_2,\theta_1-\theta_2)\,dt\\&\quad+\|p_1-p_2\|^2_{L^\infty(J;L^2(\Omega))}+\|u_1-u_2\|^2_{L^\infty(J;L^2(\Omega))}\\&\quad\leqslant C\{\|\varphi_2^1-\varphi_2^2\|^2_{L^\infty(J;H^{1/2}(\Gamma))}+\|\varphi_4^1-\varphi_4^2\|^2_{L^2(J;H^{1/2}(\Gamma))}\\&\qquad+\|s_0^1-s_0^2\|^2_{H^{-1}(\Omega)}\}.\end{aligned}$$

The result for the corresponding Newmann boundary problem is stated as follows:

Theorem 3.2 (Stability in the Newmann case). *In addition to the assumptions of Theorem* 2.6, *if* (A11) *is satisfied*, *then*

$$\begin{aligned}&\|s_1-s_2\|^2_{L^\infty(J;H^{-1}(\Omega))}+\int_J(s_1-s_2,\theta_1-\theta_2)\,dt\\&\quad+\|p_1-p_2\|^2_{L^\infty(J;L^2(\Omega))}+\|u_1-u_2\|^2_{L^\infty(J;L^2(\Omega))}\\&\quad\leqslant C\{\|\varphi_1^1-\varphi_1^2\|^2_{L^\infty(J;H^{-1/2}(\Gamma))}+\|\varphi_3^1-\varphi_3^2\|^2_{L^2(J;H^{-1/2}(\Gamma))}\\&\qquad+\|s_0^1-s_0^2\|^2_{H^{-1}(\Omega)}\}.\end{aligned}$$

Corollary (Uniqueness). *Under the assumptions of either Theorem* 3.1 *or Theorem* 3.2, *the weak solution is unique*.

3.2. Proof of Theorem 3.2

To fix the ideas, we show Theorem 3.2 in detail; the proof of Theorem 3.1 is remarked at the end of this section.

We introduce the bilinear form $a(\cdot,\cdot)$ on $H^1(\Omega)\times H^1(\Omega)$:

$$a(v,w)=(\kappa\nabla v,\nabla w)+(v,w)\quad\forall v,\ \ w\in H^1(\Omega),$$

and the Green operator $G:H^{-1}(\Omega)\to H^1(\Omega)$ by

$$a(Gv,w)=(\phi v,w)\quad\forall w\in H^1(\Omega),\ \ v\in H^{-1}(\Omega),\tag{3.1}$$

where $H^{-1}(\Omega)$ is the dual to $H^1(\Omega)$. Note that (3.1) implies

$$\|Gv\|_{H^1(\Omega)} \leqslant C\|v\|_{H^{-1}(\Omega)}. \tag{3.2}$$

Lemma 3.1. *Under the assumptions of Theorem* 2.6, *with* $t \in J$ *we have*

$$\|u_1 - u_2\|_{L^2(\Omega)} \leqslant C(\|\lambda(s_1) - \lambda(s_2)\|_{L^2(\Omega)} + \|\gamma_1(s_1) - \gamma_1(s_2)\|_{L^2(\Omega)} + \|\varphi_1^1 - \varphi_1^2\|_{H^{-1/2}(\Gamma)}).$$

Proof. It follows from (1.4) and (1.8) that

$$\begin{aligned}(\kappa\lambda(s_1)\nabla[p_1 - p_2], \nabla[p_1 - p_2]) + (\kappa[\lambda(s_1) - \lambda(s_2)]\nabla p_2, \nabla[p_1 - p_2])\\ + (\gamma_1(s_1) - \gamma_1(s_2), \nabla[p_1 - p_2]) + (\varphi_1^1 - \varphi_1^2, p_1 - p_2)_\Gamma = 0.\end{aligned} \tag{3.3}$$

Then, by the definition of V, Poincare's inequality, assumptions (A1) and (A2), and Theorem 2.6, we see that

$$\|\nabla(p_1 - p_2)\|_{L^2(\Omega)} \leqslant C(\|\lambda(s_1) - \lambda(s_2)\|_{L^2(\Omega)} + \|\gamma_1(s_1) - \gamma_1(s_2)\|_{L^2(\Omega)} + \|\varphi_1^1 - \varphi_1^2\|_{H^{-1/2}(\Gamma)}).$$

Now the desired result follows from the definition of u. □

Lemma 3.2. *Under the assumptions of Theorem* 2.6, *with* $t \in J$ *we have*

$$\begin{aligned}&\|(s_1 - s_2)(t)\|^2_{H^{-1}(\Omega)} + \int_0^t (s_1 - s_2, \theta_1 - \theta_2)\, d\tau\\ &\leqslant \varepsilon_1 \int_0^t \{\|u_1 - u_2\|^2_{L^2(\Omega)} + \|\gamma_2(s_1) - \gamma_2(s_2)\|^2_{L^2(\Omega)}\\ &\quad + \|\gamma_3(s_1) - \gamma_3(s_2)\|^2_{L^2(\Omega)}\}\, d\tau\\ &\quad + C\left\{\|s_0^1 - s_0^2\|^2_{H^{-1}(\Omega)} + \int_0^t \|\varphi_3^1 - \varphi_3^2\|^2_{H^{-1/2}(\Gamma)}\, d\tau\right\}.\end{aligned}$$

Proof. It follows from (1.5) and (1.8) that

$$\begin{aligned}&(\phi\partial_t[s_1 - s_2], G(s_1 - s_2)) + (\varphi_3^1 - \varphi_3^2, G(s_1 - s_2))_\Gamma\\ &\quad + (\kappa\nabla[\theta_1 - \theta_2], \nabla G(s_1 - s_2)) + (\kappa[\gamma_2(s_1) - \gamma_2(s_2)], \nabla G(s_1 - s_2))\\ &\quad + (u_1[\gamma_3(s_1) - \gamma_3(s_2)], \nabla G(s_1 - s_2))\\ &\quad + ([u_1 - u_2]\gamma_3(s_2), \nabla G(s_1 - s_2)) = 0.\end{aligned} \tag{3.4}$$

By (3.1), observe that

$$(\phi\partial_t[s_1 - s_2], G(s_1 - s_2)) = \tfrac{1}{2}\partial_t\|s_1 - s_2\|^2_{H^{-1}(\Omega)}.$$

Also, by (3.1), notice that

$$\begin{aligned}&(\kappa\nabla[\theta_1 - \theta_2], \nabla G(s_1 - s_2))\\ &\quad = a(\theta_1 - \theta_2, G(s_1 - s_2)) - (\theta_1 - \theta_2, G(s_1 - s_2))\\ &\quad \geqslant (\phi[\theta_1 - \theta_2], s_1 - s_2) - \|\theta_1 - \theta_2\|_{L^2(\Omega)}\|s_1 - s_2\|_{H^{-1}(\Omega)}.\end{aligned}$$

The desired result then follows from assumption (A3), (3.2), Theorem 2.6, and Gronwall's inequality. □

We now see that Theorem 3.2 follows from Lemmas 3.1 and 3.2 and assumption (A11). Remark that while we have only proven the Newmann problem, the proof for the Dirichlet case is similar. In this case, we define the bilinear form $a(\cdot,\cdot)$ on $H_0^1(\Omega)$:

$$a(v,w) = (\kappa\nabla v, \nabla w) \quad \forall v,\ w \in H_0^1(\Omega).$$

The Green operator $G : H^{-1}(\Omega) \to H_0^1(\Omega)$ is defined as before, with $H^{-1}(\Omega)$ being the dual to $H_0^1(\Omega)$. With these and an obvious modification on the boundary terms in (3.3) and (3.4) (see [14]), Theorem 3.1 can be similarly shown.

3.3. Sufficient conditions for assumption (A11)

Let η represent one of the quantities λ and γ_i $(i = 1,2,3)$. It is clear that if η satisfies that

$$|\eta(s_1) - \eta(s_2)|^2 \leqslant C(s_1 - s_2)(\theta_1 - \theta_2) \quad \forall 0 \leqslant \theta_1,\ \theta_2 \leqslant \theta^*(x), \quad \text{a.e. on } \Omega_T, \tag{3.5}$$

then assumption (A11) is true for η. A necessary and sufficient condition for (3.5) to hold is that

$$|\eta_s|^2 \leqslant Ca(s) \quad \forall s \in [0,1], \ \text{a.e. on } \Omega_T. \tag{3.6}$$

Inequality (3.6) means that η_s vanishes with a. Below we examine the conditions on η so that (3.5) or (3.6) holds.

Proposition 3.1. *Assume that* $\eta \in C^1[0,1]$, $\eta_s(0) = \eta_s(1) = 0$, η_s *is Lipschitz continuous at* 0 *and* 1, *and assumption* (A3) *is satisfied with* $0 \leqslant \alpha_1, \alpha_4 \leqslant 2$. *Then there is a constant* $C > 0$ *such that* (3.5) *holds.*

Proof. With δ as in assumption (A3), let s_1 and s_2 satisfy $0 \leqslant s_1 < \delta < 1 - \delta < s_2 \leqslant 1$. Then, by (A3), we see that

$$\theta_2 - \theta_1 = \int_{s_1}^{s_2} a(\xi)\, d\xi \geqslant \int_{\delta}^{1-\delta} a(\xi)\, d\xi \geqslant C_3(1 - 2\delta),$$

so

$$|\eta(s_2) - \eta(s_1)|^2 \leqslant \max_{0 \leqslant s \leqslant 1} \eta_s^2 (s_2 - s_1)^2 \leqslant \frac{\max_{0 \leqslant s \leqslant 1} \eta_s^2}{C_3(1 - 2\delta)} (s_2 - s_1)(\theta_2 - \theta_1).$$

That is, (3.5) is true.

We now consider the case where $0 \leqslant s_1 < \delta < s_2 \leqslant 1 - \delta$. It follows from (A3) that

$$\theta_2 - \theta_1 = \int_{s_1}^{s_2} a(\xi)\, d\xi \geqslant \int_{s_1}^{s_2} \xi^{\alpha_1}\, d\xi = \frac{1}{\alpha_1 + 1}(s_2^{\alpha_1+1} - s_1^{\alpha_1+1}). \tag{3.7}$$

Also, by the mean value theorem, there are $s_1 < \zeta_1, \zeta_2 < s_2$ such that

$$\begin{aligned} &\eta(s_2) - \eta(s_1) = \eta_s(\zeta_1)(s_2 - s_1) \text{ and} \\ &\quad s_2^{\alpha_1+1} - s_1^{\alpha_1+1} = (\alpha_1 + 1)\zeta_2^{\alpha_1}(s_2 - s_1). \end{aligned} \tag{3.8}$$

Note that

$$\zeta_2^{\alpha_1} = \frac{1}{s_2 - s_1}\int_{s_1}^{s_2} \xi^{\alpha_1}\, d\xi \geqslant \frac{1}{2}\left(\frac{s_2 + s_1}{2}\right)^{\alpha_1} \geqslant \frac{1}{2}\left(\frac{\zeta_1}{2}\right)^{\alpha_1};$$

i.e.,

$$\zeta_1 \leqslant 2^{1+\frac{1}{\alpha_1}}\zeta_2. \tag{3.9}$$

Apply (3.7)–(3.9) and the assumptions in the theorem to see that

$$\begin{aligned} |\eta(s_2) - \eta(s_1)|^2 &= \eta_s^2(\zeta_1)(s_2 - s_1)^2(\theta_2 - \theta_1)^{-1}(\theta_2 - \theta_1) \\ &\leqslant (\alpha_1 + 1)\eta_s^2(\zeta_1)(s_2^{\alpha_1+1} - s_1^{\alpha_1+1})^{-1}(s_2 - s_1)^2(\theta_2 - \theta_1) \\ &\leqslant C\zeta_1^2(\zeta_2^{\alpha_1})^{-1}(s_2 - s_1)(\theta_2 - \theta_1) \\ &\leqslant C\zeta_2^{2-\alpha_1}(s_2 - s_1)(\theta_2 - \theta_1). \end{aligned}$$

Other cases can be handled analogously. □

4. Stabilization of the weak solution

We now obtain stabilization results on the asymptotic behavior of the weak solution as $t \to \infty$. This section is independent of the last section. The results in Section 2 are needed; in particular, the uniform boundedness of s and u are utilized.

4.1. Main stabilization results

The stationary problem corresponding to (1.4), (1.5), and (1.8) is

$$\begin{aligned} &\bar{u} = -\kappa(\lambda(\bar{s})\nabla\bar{p} + \gamma_1(\bar{s})), \quad \nabla \cdot \bar{u} = 0, \qquad x \in \Omega, \\ &\phi\partial_t\bar{s} - \nabla \cdot \{\kappa(\nabla\bar{\theta} + \gamma_2(\bar{s})) + \bar{u}\gamma_3(s)\} = 0, \qquad x \in \Omega, \end{aligned} \tag{4.1}$$

with the boundary conditions

$$\begin{aligned} &\bar{u}\cdot v = \bar{\varphi}_1(x), && x\in\Gamma_1^p, \\ &\bar{p} = \bar{\varphi}_2(x), && x\in\Gamma_2^p, \\ &-\{\kappa(\nabla\bar{\theta}+\gamma_2(s))+\bar{u}\gamma_3(\bar{s})\}\cdot v = \bar{\varphi}_3(x), && x\in\Gamma_1^\theta, \\ &\bar{\theta} = \theta^*, && x\in\Gamma_2^\theta. \end{aligned} \tag{4.2}$$

Note that the phase mobility functions λ_w and λ_0 satisfy [8,10]

$$\lambda_w(0) = 0 \quad \text{and} \quad \lambda_0(1) = 0. \tag{A.12}$$

Also, if the following assumption holds:

$$\bar{\varphi}_1 = \bar{\varphi}_3, \tag{A.13}$$

by (1.6) and (A.12) we can easily see that the stationary problem in (4.1) and (4.2) has the solution $(\bar{p}, \bar{s}\equiv 1)$ where $\bar{p}$ satisfies

$$\begin{aligned} &\bar{u} = -\kappa(\lambda(1)\nabla\bar{p}+\gamma_1(1)), \quad \nabla\cdot\bar{u} = 0, && x\in\Omega, \\ &\bar{u}\cdot v = \bar{\varphi}_1(x), && x\in\Gamma_1^p, \\ &\bar{p} = \bar{\varphi}_2(x), && x\in\Gamma_2^p. \end{aligned} \tag{4.3}$$

We also have $\theta\equiv\theta^*$.

As mentioned before, system (4.3) corresponds to the physical case where the wetting phase completely displaces the nonwetting phase which initially occupied the domain Ω. Physically, that is not possible in practice because of nonzero residual saturations. Mathematically, this is possible since we can appropriately normalize the saturations.

We now show that the solution to the transient problem in (1.4), (1.5), (1.8), and (1.9) converges to the solution to the stationary problem in (4.3) as $t\to\infty$. Toward that end, we assume that

(A14) There is a constant $C>0$ such that

$$|\gamma_2(s_1)-\gamma_2(s_2)|+|\gamma_3(s_1)-\gamma_3(s_2)|\leqslant C|\theta_1-\theta_2|, \\ 0\leqslant\theta_1,\ \theta_2\leqslant\theta^*,\ s_i = \mathscr{S}(\theta_i),\ i=1,2.$$

This assumption can be remarked as for (A11) in Section 3.3. For the Dirichlet problem, we also need the assumption

$$\begin{aligned} &\varphi_2-\bar{\varphi}_2\in L^1((0,\infty);H^1(\Omega)), \quad \varphi_4-\theta^*\in L^1((0,\infty)\times\Gamma), \\ &\lim_{t\to\infty}\ \|\varphi_2-\bar{\varphi}_2\|_{H^1(\Omega)} = 0. \end{aligned} \tag{A.15}$$

Theorem 4.1 (Stabilization in the Dirichlet case). *Suppose that the assumptions of Theorem* 2.4 *and* (A.12)–(A.15) *are satisfied. Then*

$$\lim_{t\to\infty}\{\|s-1\|_{L^1(\Omega)}+\|p-\bar{p}\|_{L^2(\Omega)}+\|u-\bar{u}\|_{L^2(\Omega)}\} = 0.$$

For a corresponding result in the Newmann case, we need the assumption

$$\varphi_1 - \bar{\varphi}_1,\ \varphi_3 - \bar{\varphi}_3 \in L^1((0,\infty)\times \Gamma), \quad \lim_{t\to\infty} \|\varphi_1 - \bar{\varphi}_1\|_{L^1(\Gamma)} = 0. \quad \text{(A.16)}$$

Theorem 4.2 (Stabilization in the Newmann case). *Suppose that the assumptions of Theorem* 2.6, (A.12)–(A.14), *and* (A.16) *are satisfied. Then*

$$\lim_{t\to\infty} \{\|s-1\|_{L^1(\Omega)} + \|p-\bar{p}\|_{L^2(\Omega)} + \|u-\bar{u}\|_{L^2(\Omega)}\} = 0.$$

4.2. Proof of Theorems 4.1 and 4.2

A stabilization result was shown in [18] under abstract assumptions on the coefficients of a two-phase flow problem. In particular, when these assumptions were applied to the coefficient $a(s)$, it was required to have a degeneracy only near one:

$$a(s) = C(1-s)^{\alpha_3},$$

for $\alpha_3 \geqslant 2$. In this subsection, we prove Theorems 4.1 and 4.2 under the general assumption (A3) on $a(s)$. Moreover, the assumptions imposed in [18] are weakened here.

Lemma 4.1. *Let assumption* (A3) *be satisfied. Then, with* $\alpha_m = \max\{\alpha_i : i = 1,\dots,4\}$, *there is a positive constant C such that*

$$(s_1 - s_2)^{1+\alpha_m} \leqslant C(\theta_1 - \theta_2), \quad s_i = \mathscr{S}(\theta_i),\ i = 1,2,\ 0 \leqslant \theta \leqslant \theta^*.$$

This lemma can be shown in the same fashion as in Proposition 3.1.

Lemma 4.2. *Let the nonnegative functions* $H_1(t)$ *and* $g(t)$ *satisfy the differential inequality*

$$\frac{dg}{dt} + g^2 \leqslant H_1(t), \quad g(0) = g_0 > 0,$$

where $H_1 \in L^1(0,\infty)$. *Then we have*

$$g(t) \leqslant F(t) \equiv e^{-\int_0^t f(\tau)\,d\tau}\left(g_0 + \int_0^t e^{\int_0^\tau f(\zeta)\,d\zeta} H_1(\tau)\,d\tau\right) \to 0 \quad \textit{as } t\to\infty,$$

where the nonnegative function $f(t)$ *is the solution of the differential equation*

$$\frac{df}{dt} + f^2 = H_1(t), \quad f(0) = g_0. \tag{4.4}$$

Proof. Let $f(t)$ satisfy (4.4). Obviously, $g(t) \leqslant f(t)$ for $t \in (0,\infty)$ since $H_1(t) \geqslant 0$. Also, we can easily see that the solution f has the representation

$$f(t) = e^{-\int_0^t f(\tau)\,d\tau}\left(g_0 + \int_0^t e^{\int_0^\tau f(\zeta)\,d\zeta} H_1(\tau)\,d\tau\right).$$

These facts yield the desired result. $\square$

We now consider the auxiliary problem

$$\begin{aligned}
&\partial_t v + H(x,t)\nabla\cdot(\kappa\nabla v) + I(x,t)\cdot\nabla v = -\eta H(x,t) v && (x,t)\in\Omega_T,\\
&\kappa\nabla v\cdot\nu = 0 && (x,t)\in\Gamma_1^\theta\times J,\\
&v = 0 && (x,t)\in\Gamma_2^\theta\times J,\\
&v(x,T) = v_0(x)\geqslant 0 && x\in\Omega,
\end{aligned}\tag{4.5}$$

where $H(x,t) = H_0(x,t) + \varepsilon_1 \geqslant \varepsilon_1$, H_0, $I\in C^{\alpha,\alpha/2}(\Omega_T)$, $v_0\in C^{2+\alpha}(\Omega)$, and η is a positive constant. For this problem, we have the next result. For its proof, see Lemma 2 in [18].

Lemma 4.3. *Let $|I|\leqslant CH$ with C independent of x, t, and ε_1. Then there exists η_0 such that for all $0<\eta\leqslant\eta_0$ the solution v to (4.5) satisfies*

$$\begin{aligned}
&0\leqslant v(x,t)\leqslant \|v_0\|_{C(\Omega)} + 1 \quad (x,t)\in\Omega_T,\\
&(\kappa\nabla v,\nabla v)\leqslant C_0(T) < \infty,\\
&\left|\varepsilon_1\int_J (w,\nabla\cdot(\kappa\nabla v))\,dt\right| \leqslant \sqrt{\varepsilon_1}\,C_0(T)\|w\|_{L^2(\Omega_T)} \quad \forall w\in L^2(\Omega_T).
\end{aligned}$$

Moreover, for properly chosen v_0 we have

$$|\kappa\nabla v\cdot\nu|\leqslant C,\quad (x,t)\in\Gamma_2^\theta\times J.$$

We are now in a position to prove Theorems 4.1 and 4.2.

Proof of Theorems 4.1 and 4.2. Let v be determined by (4.5) with the coefficients given below. Without loss of generality, let $|\Omega| = 1$ and $0\leqslant v\leqslant 1$ (otherwise, consider $\Omega/|\Omega|$ and $v/\{\|v_0\|_{C(\Omega)}+1\}$ below). Then it follows from (1.5), (1.8), (4.1), and (4.2) that

$$\begin{aligned}
&\partial_t(\phi(1-s),v) - (\phi(1-s),\partial_t v) + (\kappa\nabla(\theta^* - \theta),\nabla v)\\
&\quad + (\kappa[\gamma_2(1)-\gamma_2(s)],\nabla v) + (u[\gamma_3(1)-\gamma_3(s)],\nabla v) + ([\bar u - u]\gamma_3(1),\nabla v)\\
&\quad + (\bar\varphi_3 - \varphi_3, v)_{\Gamma_1^\theta} = 0.
\end{aligned}$$

Apply Green's formula to the third term in the left-hand side of the above equation. Also, by (1.6) and (A.12), we see that $\gamma_3(1) = -1$. Consequently, by (1.4), (4.3), and (4.5), we obtain

$$\begin{aligned}
&\partial_t(\phi(1-s),v) - (\phi(1-s),\partial_t v) - (\theta^* - \theta,\nabla\cdot(\kappa\nabla v))\\
&\quad + (\theta^* - \theta,\kappa\nabla v\cdot\nu)_{\Gamma_2^\theta} + (\kappa[\gamma_2(1)-\gamma_2(s)] + u[\gamma_3(1)-\gamma_3(s)],\nabla v)\\
&\quad + (\bar\varphi_1 - \varphi_1, v)_{\Gamma_1^p} + (\bar\varphi_3 - \varphi_3, v)_{\Gamma_1^\theta} = 0.
\end{aligned}$$

That is,

$$\begin{aligned}&\partial_t(\phi(1-s),v)-(\phi(1-s),\partial_t v+H\nabla\cdot(\kappa\nabla v)+I'\nabla v)\\&\quad+\ (\theta^*-\theta,\kappa\nabla v\cdot v)_{\Gamma_2^\theta}+(\bar\varphi_1-\varphi_1,v)_{\Gamma_1^p}+(\bar\varphi_3-\varphi_3,v)_{\Gamma_1^\theta}\\&\quad+\ \varepsilon_1(\phi(1-s),\nabla\cdot(\kappa\nabla v))=0,\end{aligned}$$

where

$$\begin{aligned}&H=(\theta^*-\theta)/\{\phi(1-s)\}+\varepsilon_1,\\&I'=\{\kappa[\gamma_2(1)-\gamma_2(s)]+u[\gamma_3(1)-\gamma_3(s)]\}/\{\phi(1-s)\}.\end{aligned}$$

Denote by $\{p_h\}$ a sequence of functions that are infinitely differentiable on Ω_T such that

$$\|\nabla p_h-\nabla p\|_{L^2(\Omega)}\to 0\quad\text{as } h\to 0^+.$$

With u replaced by u_h, we define $I\equiv I'_h$ in (4.5). Now, by Lemma 4.3, we see that

$$\begin{aligned}\partial_t(\phi(1-s),v)+\eta(\theta^*-\theta,v)\leqslant\ & C\{\|I'-I'_h\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}+\|\theta^*-\varphi_4\|_{L^1(\Gamma_2^\theta)}\\&+\ \|\bar\varphi_1-\varphi_1\|_{L^1(\Gamma_1^p)}+\|\bar\varphi_3-\varphi_3\|_{L^1(\Gamma_1^\theta)}\\&+\ \sqrt{\varepsilon_1}C(T)\}.\end{aligned}$$

Letting $h\to 0$ and $\varepsilon_1\to 0$, we see that

$$\partial_t(\phi(1-s),v)+\eta(\theta^*-\theta,v)\leqslant H_1(t),\tag{4.6}$$

where

$$H_1(t)=C\{\|\theta^*-\varphi_4\|_{L^1(\Gamma_2^\theta)}+\|\bar\varphi_1-\varphi_1\|_{L^1(\Gamma_1^p)}+\|\bar\varphi_3-\varphi_3\|_{L^1(\Gamma_1^\theta)}\}.$$

By Lemma 4.1, the Jensen inequality, and the facts that $0\leqslant s\leqslant 1$ and $0\leqslant v\leqslant 1$, observe that

$$\begin{aligned}(\theta^*-\theta,v)\geqslant C((1-s)^{1+\alpha_m},v)\geqslant\ & C(1-s,v)^{1+\alpha_m}\\ \geqslant\ & C(\phi^*,\alpha_m)(\phi(1-s),v)^{\max\{2,1+\alpha_m\}}.\end{aligned}$$

Let $g=(\phi(1-s),v)$. Then (4.6) reduces to

$$\partial_t g+C(\phi^*,\alpha_m)g^{\max\{2,1+\alpha_m\}}\leqslant H_1(t),$$

with $g_0=C(\|v_0\|_{C(\Omega)}+1)$. Now, apply Lemma 4.2 to see that

$$g(T)=(\phi(1-s),v_0)(T)\leqslant F(T),$$

where $F(T)$ is given as in this lemma. Choose v_0 in (4.5) such that $v_0^{-1} \in L^{\pi}(\Omega)$, $0 < \pi < 1$. Then it follows from the Hölder inequality that

$$\|1 - s\|_{L^1(\Omega)} \leqslant C\{\|v_0^{-1}\|_{L^{\pi}(\Omega)} F(T)\}^{\pi/(\pi+1)} \to 0 \quad \text{as } T \to \infty.$$

Apply Poincare's inequality to have

$$\|p - \bar{p}\|_{L^2(\Omega)} \leqslant C(\|\nabla(p - \bar{p})\|_{L^2(\Omega)} + \|\varphi_2 - \bar{\varphi}_2\|_{H^1(\Omega)}).$$

Also, as in Lemma 3.1, by the uniform boundedness of p and ∇p we have

$$\|\nabla(p - \bar{p})\|_{L^2(\Omega)} \leqslant C(\|1 - s\|_{L^1(\Omega)} + \|\varphi_1 - \bar{\varphi}_1\|_{L^1(\Gamma_1^p)}).$$

Therefore, the theorem follows. □

5. An example

In this section, we present an example to show typical regularity of the saturation. Namely, we consider the so-called porous medium equation

$$\partial_t s - \Delta s^m = 0, \quad m > 1.$$

This equation can be equivalently rewritten in form (1.5):

$$\partial_t s - \nabla \cdot (m s^{m-1} \nabla s) = 0, \quad m > 1, \tag{5.1}$$

so we see that the diffusion coefficient $a(s)$ is ms^{m-1} and the variable θ equals s^m. Obviously, (5.1) is degenerate at zero. Note that (1.5) reduces to (5.1) if $\gamma_2 = \gamma_3 = 0$ and $\phi = 1$. Also, Eq. (5.1) often arises in the flow of a gas in porous media. To see this, ignoring certain constants, the gas flow is governed by

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad v = -\nabla p, \quad \rho = p^{\gamma}, \tag{5.2}$$

where ρ is the density, p the pressure, v the velocity, and γ a (constant) ratio of specific heats. These equations are the mass conservation, Darcy's law, and equation of state [8,25], respectively. Eliminating v and p in (5.2), we obtain

$$\partial_t \rho - \frac{1}{1+\gamma} \Delta(\rho^{1+1/\gamma}) = 0.$$

Rescaling t by $1/(1+\gamma)$ leads to (5.1) with $s = \rho$. Hence we see that one of the advantages of writing the two-phase flow equations (1.1) in (1.4) and (1.5) is that the analysis also applies to the single-phase flow.

Beginning from a delta function of integral Γ at the original, the exact solution to (5.1) is of the form [7,24]

$$u(|x|, t) = \max\left\{0, t^{-\alpha}\left(\Gamma - \frac{\alpha(m-1)}{2dm} \frac{|x|^2}{t^{2\alpha/d}}\right)^{1/(m-1)}\right\},$$

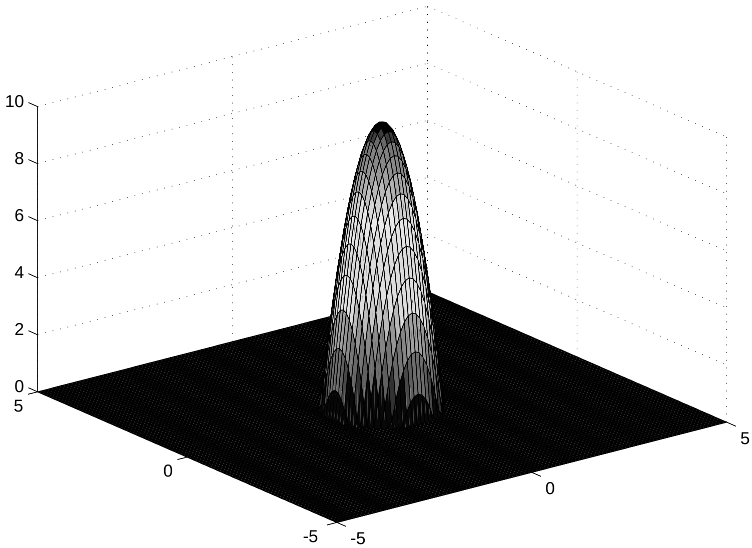

Fig. 1.

where $\alpha = 1/(m - 1 + 2/d)$. Fig. 1 shows an example of this solution in two dimensions. It is radially symmetric and has compact support. Also, the solution contains an interface where the gradient is discontinuous. With the present choice of the initial datum, (5.1) corresponds to the flow case with a point source.

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