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THE MULTISCALE FINITE ELEMENT METHOD WITH NONCONFORMING ELEMENTS FOR ELLIPTIC HOMOGENIZATION PROBLEMS*

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Abstract. The multiscale finite element method was developed by Hou and Wu [*J. Comput. Phys.*, 134 (1997), pp. 169–189] to capture the effect of microscales on macroscales for multiscale problems through modification of finite element basis functions. For second-order multiscale partial differential equations, continuous (conforming) finite elements have been considered so far. Efendiev, Hou, and Wu [*SIAM J. Numer. Anal.*, 37 (2000), pp. 888–910] considered a nonconforming multiscale finite element method where nonconformity comes from an oversampling technique for reducing resonance errors. In this paper we study the multiscale finite element method in the context of nonconforming finite elements for the first time. When the oversampling technique is used, a double nonconformity arises: one from this technique and the other from nonconforming elements. An equivalent formulation recently introduced by Chen [*Numer. Methods Partial Differential Equations*, 22 (2006), pp. 317–360] (also see [Y. R. Efendiev, T. Hou, and V. Ginting, *Commun. Math. Sci.*, 2 (2004), pp. 553–589]) for the multiscale finite element method, which utilizes standard basis functions of finite element spaces but modifies the bilinear (quadratic) form in the finite element formulation of the underlying multiscale problems, is employed in the present study. Nonlinear multiscale and random homogenization problems are also studied, and numerical experiments are presented.

Key words. multiscale problem, multiscale finite element method, finite element, nonconforming finite element, oversampling technique, convergence, stability, error estimate, nonlinear problem, random problem

AMS subject classifications. 35K60, 35K65, 76S05, 76T05

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1. Introduction.

Hou and Wu [18] introduced the multiscale finite element method for numerical solution of multiscale problems that are described by partial differential equations with highly oscillatory coefficients. The main idea of this method is to incorporate the microscale information of a multiscale differential problem into finite element basis functions. It is through these modified bases and finite element formulations that the effect of microscales on macroscales can be correctly captured.

A convergence analysis of the method was given in [19] for a two-scale homogenization problem with periodic coefficients. It was proven that the multiscale finite element solution converges to the homogenized solution as $h, \epsilon \to 0$, where h is the mesh size and ϵ is the small scale in the solution. The analysis also indicated that a resonance error exists between the grid scale and the scales of the homogenization

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problem. This is a common feature in some numerical upscaling techniques. This error represents a mismatch between the local construction of the multiscale basis functions and the global nature of the continuous problem. An oversampling technique was analyzed in [16] to reduce the resonance error. The idea of this technique is to construct the local basis functions over a domain with size larger than h to reduce the boundary layer effect present in the first-order corrector of the local solution. The oversampling technique results in a nonconforming multiscale method.

The multiscale finite element method and its oversampled version have been based on continuous (conforming) basis functions for second-order partial differential equations. In this paper we study this method and its variations using nonconforming finite elements. The nonconforming elements have been widely used in computational mechanics and structural engineering due to the fact that they employ fewer degrees of freedom than their conforming counterparts for the same partial differential problem. However, a convergence analysis for the nonconforming elements is more difficult because of nonconformity. In fact, there is no general convergence theory available for these elements [6]. If the oversampling technique is used in the setting of the nonconforming elements, a double nonconformity arises. Here we show how the convergence results obtained for the conforming multiscale method can be extended to the nonconforming multiscale method.

The present analysis is based on an equivalent formulation recently introduced by Chen [7] (also see [15]) for the multiscale finite element method, which utilizes standard basis functions of finite element spaces but modifies the bilinear (quadratic) form in the finite element formulation of the underlying multiscale problems. This new formulation captures the macroscale structure of the solution of a differential multiscale problem through the modification of this bilinear form. It is a general approach that can handle a large variety of differential problems, periodic or nonperiodic, linear or nonlinear, and stationary or dynamic, and can be applied in a variety of finite elements, conforming or nonconforming, and Galerkin or mixed, as shown here.

The paper is organized as follows. In the next section we present a continuous two-scale problem and the traditional nonconforming finite element method for it. The multiscale finite element method using nonconforming basis functions is defined in the third section. A homogenization theory is reviewed in the fourth section; homogenization is used only in the convergence analysis. The convergence analysis for the cases $h \ll \epsilon$ and $\epsilon \ll h$ is given in the fifth section. The extensions of the nonconforming multiscale finite element method and its analysis to random and nonlinear homogenization problems are described in the sixth and seventh sections, respectively. Finally, numerical experiments are given in the last section. As a general remark, the generic constant $C > 0$ is assumed to be independent of the mesh size h and the microscale ϵ throughout this paper.

2. Preliminaries. Let Ω be a bounded domain in $\mathbb{R}^d$, $1 \le d \le 3$, with Lipschitz boundary Γ. For a subdomain $D \subset \Omega$, each integer $m \ge 0$, and each real number $1 \le p \le \infty$, $W^{m,p}(D)$ indicates the usual Sobolev space of real functions that have all of their weak derivatives of order up to m in the Lebesgue space $L^p(D)$. The norm and seminorm of $W^{m,p}(D)$ are denoted by $\|\cdot\|_{m,p,D}$ and $|\cdot|_{m,p,D}$, respectively. When $p = 2$, $W^{m,p}(D)$ is written as $H^m(D)$ with the norm $\|\cdot\|_{m,D}$ and the seminorm $|\cdot|_{m,D}$. We also use the space

$$H_0^1(D) = \{v \in H^1(D) : v|_{\partial D} = 0\}.$$

We consider the second-order elliptic problem

$$(2.1)\qquad \begin{aligned} -\nabla\cdot\left(a_\epsilon \nabla u^\epsilon\right) &= f && \text{in } \Omega,\\ u^\epsilon &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in L^2(\Omega)$ is a given function and $a_\epsilon = (a_{ij}(x/\epsilon))$ is a symmetric, positive definite, bounded tensor:

$$(2.2)\qquad a_*|\zeta|^2 \le \sum_{i,j=1}^{d} a_{ij}(y)\zeta_i\zeta_j \le a^*|\zeta|^2 \qquad \forall y,\ \zeta \in \mathbb{R}^d,$$

for some positive constants a_* and a^*. In the first five sections we assume that $a(y)$ is smooth and periodic in y with period $I = [0,1]^d$. In problem (2.1), the multiscale feature is reflected in the oscillatory nature of the coefficient a_ϵ for $\epsilon \ll 1$, which represents the microscale. For simplicity, we consider the homogeneous Dirichlet boundary condition in (2.1). Also, the subsequent methods and their analysis can be given when a_ϵ is of the form $(a_{ij}(x, x/\epsilon))$ (the locally periodic case) [9, 16] (see the seventh section).

Let $U = H_0^1(\Omega)$. The variational form of (2.1) is to find $u^\epsilon \in U$ such that

$$(2.3)\qquad (a_\epsilon \nabla u^\epsilon, \nabla v) = (f, v) \qquad \forall v \in U,$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^d$, as appropriate.

For $h > 0$, let T_h be a regular, quasi-uniform macroscale partition of Ω [6, 11], where the mesh size h resolves the variations of Ω, f, and the slow variable of a_ϵ. Associated with T_h, let $U_h \subset L^2(\Omega)$ be a finite element space such that for any $v \in U \cap H^l(\Omega)$ there exists a $v_h \in U_h$ satisfying the approximation property

$$(2.4)\qquad \|v - v_h\|_{0,\Omega} + h|v - v_h|_{1,h} \le Ch^l|v|_{l,\Omega}, \qquad 1 \le l \le r+1,$$

where $|v|_{1,h} = (\sum_{T\in T_h} |v|_{1,T}^2)^{1/2}$. The traditional (nonconforming) finite element method for (2.1) is to seek $u_h \in U_h$ such that

$$(2.5)\qquad \sum_{T\in T_h} (a_\epsilon \nabla u_h, \nabla v)_T = (f, v) \qquad \forall v \in U_h.$$

A nonconforming finite element analysis [6, 11] shows that the error estimate holds:

$$(2.6)\qquad \|u^\epsilon - u_h\|_{0,\Omega} + h|u^\epsilon - u_h|_{1,h} \le Ch^l|u^\epsilon|_{l,\Omega}, \qquad 1 \le l \le r+1.$$

Because the following regularity result for (2.1) holds [21],

$$(2.7)\qquad |u^\epsilon|_{2,\Omega} \le C\epsilon^{-1}\|f\|_{0,\Omega},$$

estimate $|u^\epsilon - u_h|_{1,h}$ in (2.6) deteriorates for small ϵ. One of the aims of this paper is to introduce multiscale methods to derive improved error estimates.

3. The MsFEM. As discussed in the introduction, the multiscale finite element method (MsFEM) developed in [3, 18] uses modified basis functions. In this section, following [7], we define the MsFEM in an equivalent form, which utilizes standard basis functions but modifies the bilinear form in (2.5).

For any $v \in U_h$, we define $R_T(v) \in H^1(T)$, $T \in T_h$, by

$$(3.1)\qquad \begin{aligned} &(a_\epsilon \nabla R_T(v), \nabla w)_T = 0 && \forall w \in H_0^1(T),\\ &R_T(v) = v && \text{on } \partial T. \end{aligned}$$

The global operator R is then given by

$$R(v)|_T = R_T(v) \qquad \forall v \in U_h,\ T \in T_h.$$

It is easy to see that $R(v) \notin U$, $v \in U_h$. Now the MsFEM for (2.1) is to seek $u_h \in U_h$ such that

$$(3.2)\qquad \sum_{T\in T_h} (a_\epsilon \nabla R(u_h), \nabla R(v))_T = (f, R(v)) \qquad \forall v \in U_h.$$

Note that the major difference between (2.5) and (3.2) lies in the modification of the bilinear form, which needs the solution of local problems (3.1). It is through these local problems and the finite element formulation that the effect of microscales on macroscales can be correctly captured. Since these local problems are independent of each other, they can be solved in parallel.

We will give a convergence analysis for (3.2). Throughout this paper we will perform all proofs in detail for the lowest-order nonconforming finite element space on triangles (respectively, simplices) [6, 12]. We point out that there is no technical difficulty in extending all arguments to spaces of higher order and other types of nonconforming finite elements [1, 6]. In the lowest-order case, the nonconforming space U_h is

$U_h = \{v \in L^2(\Omega) : v|_T$ is linear, $T \in T_h$; v is continuous at the midpoints of interior edges (respectively, centroids of interior faces) and is zero at the midpoints of edges (respectively, centroids of faces) on $\Gamma\}$.

The existence and uniqueness of a solution to (3.2) can be shown as in the conforming MsFEM [7]. Moreover, the following equivalence holds:

$$(3.3)\qquad C_1|v|_{1,T} \le |R_T(v)|_{1,T} \le C_2|v|_{1,T} \qquad \forall v \in U_h,$$

and the solution u_h satisfies the stability result

$$(3.4)\qquad |u_h|_{1,h} + |R(u_h)|_{1,h} \le C\|f\|_{0,\Omega}.$$

4. Homogenization theory. The convergence analysis for the case $\epsilon \ll h$ will be different from that for $h \ll \epsilon$ and will utilize a macroscopic model of (2.1). Here we collect some results from the homogenization theory. The homogenized problem of (2.1) reads as follows: Find $U_0 \in U$ such that

$$(4.1)\qquad (A\nabla U_0, \nabla v) = (f, v) \qquad \forall v \in U,$$

where the homogenized matrix $A = (A_{ij})$ is given by

$$A_{ij} = \frac{1}{|I|}\int_I \left(a_{ij}(y) + \sum_{k=1}^{d} \left(a_{ik}\frac{\partial \chi^j}{\partial y_k}\right)(y)\right)\, dy$$

and χ^j satisfies, with a periodic boundary condition in y,

$$-\nabla_y \cdot \left(a_\epsilon \nabla_y \chi^j\right) = \sum_{i=1}^{d} \frac{\partial}{\partial y_i} a_{ij}(y), \quad y \in I, \quad \int_I \chi^j(y)\, dy = 0. \tag{4.2}$$

It is well known that A is symmetric and positive definite. We will assume that $\chi^j \in W^{1,\infty}(I)$, which is true if $a_{ij} \in W^{1,\ell}(I)$, $\ell > 2$ [21].

Define

$$u_1^\epsilon = U_0 + \epsilon \sum_{k=1}^{d} \chi^k \frac{\partial U_0}{\partial x_k}. \tag{4.3}$$

Then simple algebraic manipulations give

$$\sum_{j=1}^{d} a_{ij}\left(\frac{x}{\epsilon}\right) \frac{\partial u_1^\epsilon}{\partial x_j} = \sum_{j=1}^{d} \left(A_{ij} + g_{ij}\left(\frac{x}{\epsilon}\right)\right) \frac{\partial U_0}{\partial x_j} + \epsilon \sum_{j=1}^{d} \left(a_{ij}\chi^k\right)\left(\frac{x}{\epsilon}\right) \frac{\partial^2 U_0}{\partial x_k \partial x_j}, \tag{4.4}$$

where

$$g_{ij}(y) = a_{ij}(y) + \sum_{l=1}^{d} \left(a_{il} \frac{\partial \chi^j}{\partial y_l}\right)(y) - A_{ij}. \tag{4.5}$$

In matrix form, (4.4) is given by

$$a_\epsilon \nabla u_1^\epsilon = A \nabla U_0 + g \nabla U_0 + \epsilon \sum_{k=1}^{d} a_\epsilon \chi_k \cdot \nabla \frac{\partial U_0}{\partial x_k}, \tag{4.6}$$

where $g = (g_{ij})$.

Note that g_{ij} is periodic in y and

$$\int_I g_{ij}(y)\, dy = 0,$$

by the definition of A. Also, note that

$$\sum_{i=1}^{d} \frac{\partial}{\partial y_i} g_{ij}(y) = 0.$$

Hence there is a skew-symmetric matrix $\alpha^j(y)$ such that

$$g_{ij} = \sum_{k=1}^{d} \frac{\partial}{\partial y_k} \alpha_{ik}^j(y), \quad \int_I \alpha_{ik}^j(y)\, dy = 0.$$

Therefore, we see that

$$\sum_{j=1}^{d} g_{ij}\left(\frac{x}{\epsilon}\right) \frac{\partial U_0}{\partial x_j} = \epsilon \sum_{j,k=1}^{d} \left\{ \frac{\partial}{\partial x_k} \left(\alpha_{ik}^j\left(\frac{x}{\epsilon}\right) \frac{\partial U_0}{\partial x_j} \right) - \alpha_{ik}^j\left(\frac{x}{\epsilon}\right) \frac{\partial^2 U_0}{\partial x_j \partial x_k} \right\}. \tag{4.7}$$

Equations (4.3)–(4.7), together with (4.8) and (4.9) below, will be employed in the subsequent analysis. It follows from [9, 25] that there is a constant C, independent of ϵ, such that

$$\|u^\epsilon - U_0\|_{0,\Omega} \leq C\epsilon \left(|U_0|_{2,\Omega} + |U_0|_{1,\Omega}\right). \tag{4.8}$$

The following lemma extends a similar result in [25]. Also, an analogous result was shown for a Neumann problem in [9]. The proof for the Dirichlet problem (2.1) is different from that for the Neumann problem [7].

LEMMA 4.1. *Let $U_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ and u_1^ϵ be defined by* (4.3)*. Then there is a constant C, independent of ϵ and Ω, such that*

$$|u^\epsilon - u_1^\epsilon|_{1,\Omega} \leq C\left\{\epsilon|U_0|_{2,\Omega} + \sqrt{\epsilon|\Gamma|}\ |U_0|_{1,\infty,\Omega}\right\}, \tag{4.9}$$

where $|\Gamma|$ is the length of Γ ($d = 2$) or its area ($d = 3$).

5. Convergence analysis.

For the nonconforming finite element method considered, it is known that the Céa lemma is no longer valid. Strang's second lemma [23] can be easily shown by using assumption (2.2) and (3.2) [6, 11].

LEMMA 5.1. *Let u^ϵ and u_h be the respective solutions of* (2.3) *and* (3.2)*. Then there is a constant $C > 0$, independent of h and ϵ, such that*

$$\begin{aligned}|u^\epsilon - R(u_h)|_{1,h} \leq C\Bigg\{ &\inf_{v\in U_h} |u^\epsilon - R(v)|_{1,h} \\ &+ \sup_{w\in U_h, w\neq 0} \frac{\left|\sum_{T\in T_h}\left[\left(a_\epsilon\nabla u^\epsilon, \nabla R_T(w)\right)_T - (f, R_T(w))_T\right]\right|}{|w|_{1,h}}\Bigg\}.\end{aligned} \tag{5.1}$$

In (5.1), the first term in the right-hand side is referred to as the approximation error, and the second term is called the consistency error. The latter error stems from nonconformity.

5.1. The case $h \ll \epsilon$.

In the case $h \ll \epsilon$, the traditional nonconforming finite element method and the MsFEM behave similarly. In fact, the error bound (2.6) holds for (3.2). For completeness, we show the error estimate for this case.

THEOREM 5.2. *Let u^ϵ and u_h be the respective solutions of* (2.3) *and* (3.2)*. Then there is a constant $C > 0$, independent of h and ϵ, such that*

$$\|u^\epsilon - R(u_h)\|_{0,\Omega} + h|u^\epsilon - R(u_h)|_{1,h} \leq \frac{Ch^2}{\epsilon}\|f\|_{0,\Omega}. \tag{5.2}$$

Proof. Define the conforming finite element space

$$V_h = \{v \in U :\ v|_T \text{ is linear, } T \in T_h\}.$$

Because $V_h \subset U_h$, it follows from Theorem 2.5 in [7] that the approximation error in (5.1) can be estimated as follows:

$$\inf_{v\in U_h} |u^\epsilon - R(v)|_{1,h} \leq \frac{Ch^2}{\epsilon}\|f\|_{0,\Omega}. \tag{5.3}$$

It thus suffices to bound the consistency error in (5.1). By using Green's formula and (2.1), for $w \in U_h$ we see that

$$\begin{aligned}\sum_{T\in T_h} &\left[\left(a_\epsilon \nabla u^\epsilon, \nabla R_T(w)\right)_T - (f, R_T(w))_T\right] \\ &= \sum_{T\in T_h} \left[\left(a_\epsilon \nabla u^\epsilon \cdot \nu, R_T(w)\right)_{\partial T} - \left(\nabla\cdot(a_\epsilon\nabla u^\epsilon), R_T(w)\right)_T - (f, R_T(w))_T\right] \\ &= \sum_{T\in T_h} \left(a_\epsilon \nabla u^\epsilon \cdot \nu, R_T(w)\right)_{\partial T},\end{aligned}$$

where ν is the outward unit normal to ∂T. Thus, by the definition of $R_T(w)$ in (3.1) (i.e., $R_T(w) = w$ on ∂T),

$$\sum_{T\in T_h} \left[\left(a_\epsilon \nabla u^\epsilon, \nabla R_T(w)\right)_T - (f, R_T(w))_T\right] = \sum_{T\in T_h} \left(a_\epsilon \nabla u^\epsilon \cdot \nu, w\right)_{\partial T}. \tag{5.4}$$

Consequently, application of the standard convergence argument for the nonconforming finite element method under consideration [6, 11] to (5.4) yields

$$\left|\sum_{T\in T_h} \left[\left(a_\epsilon \nabla u^\epsilon, \nabla R_T(w)\right)_T - (f, R_T(w))_T\right]\right| \le Ch|u^\epsilon|_{2,\Omega}|w|_{1,h}. \tag{5.5}$$

Therefore, combine (2.7), (5.1), (5.3), and (5.5) to obtain the desired result (5.2) for the norm $|\cdot|_{1,h}$. A standard duality argument for the nonconforming method [6, 11] can be applied to obtain the L^2-estimate in (5.2). ∎

5.2. The case $\epsilon \ll h$. The convergence analysis for the case $\epsilon \ll h$ is very different from that for $h \ll \epsilon$. In the present case, the homogenization theory in the fourth section will be used.

For any $w \in U_h$, define

$$Q(w) = w + \epsilon \sum_{k=1}^{d} \chi^k \frac{\partial w}{\partial x_k}, \tag{5.6}$$

where the function χ^k is defined in (4.2), $k = 1, 2, \ldots, d$.

LEMMA 5.3. *Let $R(w)$ and $Q(w)$ be defined by* (3.1) *and* (5.6)*, respectively. Then*

$$\|R(w) - Q(w)\|_{0,T} \le C\epsilon h_T^{(d-1)/2}|w|_{1,\infty,T}, \qquad T\in T_h,\ w\in U_h, \tag{5.7}$$

and

$$|R(w) - Q(w)|_{1,T} \le C\epsilon h_T^{d/2-1}|w|_{1,\infty,T}, \qquad T\in T_h,\ w\in U_h. \tag{5.8}$$

Proof. On each $T \in T_h$, we define a boundary corrector θ_ϵ by

$$\begin{aligned} -\nabla\cdot\left(a_\epsilon \nabla\theta_\epsilon\right) &= 0 \qquad \text{in } T, \\ \theta_\epsilon &= -\epsilon\sum_{k=1}^{d}\chi^k\frac{\partial w}{\partial x_k} \qquad \text{on } \partial T. \end{aligned} \tag{5.9}$$

It can be checked that

$$a_\epsilon \nabla Q(w) = A\nabla w + g\nabla w, \qquad T \in T_h. \tag{5.10}$$

By combining (3.1), (5.9), and (5.10), we see that

$$R(w) - Q(w) = \theta_\epsilon, \qquad x \in T. \tag{5.11}$$

Thus it suffices to estimate θ_ϵ.

Note that

$$\|\theta_\epsilon\|_{0,T} \le C\epsilon \left\| \sum_{k=1}^{d} \chi^k \frac{\partial w}{\partial x_k} \right\|_{0,\partial T} \le C\epsilon h_T^{(d-1)/2} |w|_{1,\infty,T},$$

which gives (5.7). Next, applying a maximum principle to (5.9) yields [21]

$$\|\theta_\epsilon\|_{0,\infty,T} \le C\epsilon |w|_{1,\infty,T}. \tag{5.12}$$

Also, an interior estimate [2] implies that

$$|\theta_\epsilon|_{1,T} \le C h_T^{-1} \|\theta_\epsilon\|_{0,T},$$

which, together with (5.12), gives

$$|\theta_\epsilon|_{1,T} \le C\epsilon h_T^{d/2-1} |w|_{1,\infty,T}. \tag{5.13}$$

Finally, by (5.11) and (5.13), the desired result (5.8) follows. ▯

The proof of the next lemma can be found in [9].

LEMMA 5.4. *Let $v \in L^\infty(\mathbb{R}^d)$ ($d = 2$ or 3) be a periodic function with respect to I and its average over I be zero. Then, for any $w \in H^1(T) \cap L^\infty(T)$, $T \in T_h$,*

$$\left| \int_T v\left(\frac{x}{\epsilon}\right) w(x)\, dx \right| \le C\epsilon \left\{ h_T^{d/2} |w|_{1,T} + h_T^{d-1} \|w\|_{0,\infty,T} \right\}. \tag{5.14}$$

We now derive error estimates for the case $\epsilon \ll h$.

THEOREM 5.5. *Assume that $U_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$. If $u^\epsilon \in U$ and $u_h \in U_h$ are the respective solutions of* (2.3) *and* (3.2), *then the error estimates hold:*

$$\begin{aligned} |u^\epsilon - R(u_h)|_{1,\Omega} &\le C\left\{ (h+\epsilon)\|f\|_{0,\Omega} + \sqrt{\frac{\epsilon}{h}}\ |U_0|_{1,\infty,\Omega} \right\}, \\ \|u^\epsilon - R(u_h)\|_{0,\Omega} &\le C\left\{ (h^2+\epsilon)\|f\|_{0,\Omega} + \sqrt{\frac{\epsilon}{h}}\ |U_0|_{1,\infty,\Omega} \right\}. \end{aligned} \tag{5.15}$$

Proof. Again, since the conforming finite element space V_h is a subspace of U_h, it follows from Theorem 2.9 in [7] that the approximation error in (5.1) can be estimated by

$$\inf_{v \in U_h} |u^\epsilon - R(v)|_{1,h} \le C\left\{ (h+\epsilon)\|f\|_{0,\Omega} + \sqrt{\frac{\epsilon}{h}}\ |U_0|_{1,\infty,\Omega} \right\}, \tag{5.16}$$

so it is sufficient to estimate the consistency error in (5.1).

By (5.6), for $w \in U_h$ we see that

$$\begin{aligned} &\left(a_\epsilon \nabla u^\epsilon, \nabla R_T(w)\right)_T - (f, R_T(w))_T \\ &\quad = \left(a_\epsilon \nabla u^\epsilon, \nabla[R_T(w) - Q(w)]\right)_T - (f, R_T(w) - Q(w))_T + \left(a_\epsilon \nabla u^\epsilon, \nabla w\right)_T - (f, w)_T \\ &\qquad + \left(a_\epsilon \nabla u^\epsilon, \epsilon \sum_{k=1}^{d} \nabla \chi^k \frac{\partial w}{\partial x_k}\right)_T - \left(f, \epsilon \sum_{k=1}^{d} \chi^k \frac{\partial w}{\partial x_k}\right)_T. \end{aligned} \tag{5.17}$$

We estimate each term in the right-hand side of (5.17). First, by using (5.8), we see that

$$|\left(a_\epsilon \nabla u^\epsilon, \nabla[R_T(w) - Q(w)]\right)_T| \leq C\epsilon h_T^{d/2-1} |u^\epsilon|_{1,T} |w|_{1,\infty,T}. \tag{5.18}$$

Also, by applying (5.7), we have

$$|(f, R_T(w) - Q(w))_T| \leq C\epsilon h_T^{(d-1)/2} \|f\|_{0,T} |w|_{1,\infty,T}. \tag{5.19}$$

Next, by using relation (4.6), we write

$$\begin{aligned}\left(a_\epsilon \nabla u^\epsilon, \nabla w\right)_T - (f, w)_T &= (a_\epsilon \nabla[u^\epsilon - u_1^\epsilon], \nabla w)_T + (a_\epsilon \nabla u_1^\epsilon, \nabla w)_T - (f, w)_T \\ &= (a_\epsilon \nabla[u^\epsilon - u_1^\epsilon], \nabla w)_T + (A\nabla U_0, \nabla w)_T - (f, w)_T \\ &\quad + (g\nabla U_0, \nabla w)_T + \epsilon \left(\sum_{k=1}^d a_\epsilon \chi_k \cdot \nabla \frac{\partial U_0}{\partial x_k}, \nabla w \right)_T.\end{aligned} \tag{5.20}$$

It is clear that

$$|(a_\epsilon \nabla[u^\epsilon - u_1^\epsilon], \nabla w)_T| \leq C|u^\epsilon - u_1^\epsilon|_{1,T} |w|_{1,T}. \tag{5.21}$$

Application of the standard convergence argument for the nonconforming finite element method [6] gives

$$\left| \sum_{T \in T_h} \left[(A\nabla U_0, \nabla w)_T - (f, w)_T \right] \right| \leq Ch|U_0|_{2,\Omega} |w|_{1,h}. \tag{5.22}$$

It follows from (5.14) that

$$|(g\nabla U_0, \nabla w)_T| \leq C\epsilon \left\{ h_T^{d/2} |U_0|_{2,T} + h_T^{d-1} \|U_0\|_{1,\infty,T} \right\} |w|_{1,\infty,T}. \tag{5.23}$$

Obviously, it holds that

$$\left| \epsilon \left(\sum_{k=1}^d a_\epsilon \chi_k \cdot \nabla \frac{\partial U_0}{\partial x_k}, \nabla w \right)_T \right| \leq C\epsilon |U_0|_{2,T} |w|_{1,T}. \tag{5.24}$$

Now, substituting (5.21)–(5.24) into (5.20) implies that

$$\begin{aligned}&\left| \sum_{T \in T_h} \left[\left(a_\epsilon \nabla u^\epsilon, \nabla w\right)_T - (f, w)_T \right] \right| \\ &\leq C \Bigg\{ \sum_{T \in T_h} \left(|u^\epsilon - u_1^\epsilon|_{1,T} |w|_{1,T} + \epsilon \left(h_T^{d/2} |U_0|_{2,T} + h_T^{d-1} |U_0|_{1,\infty,T} \right) |w|_{1,\infty,T} \right) \\ &\qquad + (h + \epsilon)|U_0|_{2,\Omega} |w|_{1,h} \Bigg\}.\end{aligned} \tag{5.25}$$

Again, by using relation (4.6), we write

$$\left(a_\epsilon \nabla u^\epsilon, \epsilon \sum_{k=1}^{d} \nabla \chi^k \frac{\partial w}{\partial x_k}\right)_T$$
$$= \left(a_\epsilon \nabla[u^\epsilon - u_1^\epsilon], \sum_{k=1}^{d} \nabla_y \chi^k \frac{\partial w}{\partial x_k}\right)_T + \left(a_\epsilon \nabla u_1^\epsilon, \sum_{k=1}^{d} \nabla_y \chi^k \frac{\partial w}{\partial x_k}\right)_T$$
$$= \left(a_\epsilon \nabla[u^\epsilon - u_1^\epsilon], \sum_{k=1}^{d} \nabla_y \chi^k \frac{\partial w}{\partial x_k}\right)_T + \left(A \nabla U_0, \sum_{k=1}^{d} \nabla_y \chi^k \frac{\partial w}{\partial x_k}\right)_T$$
$$+ \left(g \nabla U_0, \sum_{k=1}^{d} \nabla_y \chi^k \frac{\partial w}{\partial x_k}\right)_T + \epsilon \left(\sum_{k=1}^{d} a_\epsilon \chi_k \cdot \nabla \frac{\partial U_0}{\partial x_k}, \sum_{k=1}^{d} \nabla_y \chi^k \frac{\partial w}{\partial x_k}\right)_T,$$

and an argument similar to that for (5.25) yields

$$\begin{aligned} &\left|\left(a_\epsilon \nabla u^\epsilon, \epsilon \sum_{k=1}^{d} \nabla \chi^k \frac{\partial w}{\partial x_k}\right)_T\right| \\ &\le C\Big\{ |u^\epsilon - u_1^\epsilon|_{1,T} |w|_{1,T} + \epsilon |U_0|_{2,T} |w|_{1,T} \\ &\quad + \epsilon \left(h_T^{d/2} |U_0|_{2,T} + h_T^{d-1} |U_0|_{1,\infty,T}\right) |w|_{1,\infty,T} \Big\}. \end{aligned} \tag{5.26}$$

Finally, it is easy to see that

$$\left|\left(f, \epsilon \sum_{k=1}^{d} \chi^k \frac{\partial w}{\partial x_k}\right)_T\right| \le C\epsilon \|f\|_{0,T} |w|_{1,T}. \tag{5.27}$$

Now, combine (5.16)–(5.19), (5.25)–(5.27), and Lemmas 4.1 and 5.1 to obtain the desired result in the norm $|\cdot|_{1,h}$ in (5.15). A duality argument in the nonconforming setting [6, 11] can be employed to obtain the L^2-estimate in (5.15). ∎

5.3. An oversampling technique. Note that estimates (5.15) deteriorate when ϵ is of the same order as the mesh size h. This phenomenon reveals a "resonance error" between the grid scale h and the scale ϵ of the continuous problem (2.1). The resonance is due to a mismatch between the local solution of (3.1) and the global solution of (2.1) on the boundary of each $T \in T_h$, which produces a boundary layer. Since this layer is thin, we can sample in a (local) domain with size larger than h and utilize only the interior sampled information. In this manner, the influence of the boundary layer in the larger domain can be greatly reduced. In this subsection, we extend this technique for the conforming MsFEM [16, 18] to the MsFEM in the present nonconforming finite element setting in order to reduce the resonance error in (5.15).

For each $T \in T_h$, we indicate by $S(T)$ a macroelement which contains T and satisfies the following condition: There are positive constants C_1 and C_2, independent of h and ϵ, such that $h_S \le C_1 h_T$ and $\operatorname{dist}(\partial S, \partial T) \ge C_2 h_T$, where h_S is the diameter of S. For each $v \in U_h(T)$ (the restriction of U_h to T), we extend it to $U_h(S)$ as follows. Let $\{\phi_i^T\}_1^{d+1}$ and $\{\psi_i^S\}_1^{d+1}$ be the respective bases of $U_h(T)$ and $U_h(S)$. Set

$$v|_T = \sum_{i=1}^{d+1} c_i^T \phi_i^T, \quad \phi_i^T = \sum_{j=1}^{d+1} c_{ij}^T \psi_j^S|_T.$$

Then we define $\widehat{v} \in U_h(S)$ by

$$\widehat{v} = \sum_{i,j=1}^{d+1} c_i^T c_{ij}^T \psi_j^S.$$

Now, for any $v \in U_h$, we define $R_S(v) \in H^1(S)$, $T \subset S$, $T \in T_h$, by

$$\begin{aligned} &(a_\epsilon \nabla R_S(v), \nabla w)_S = 0 \qquad && \forall w \in H_0^1(S), \\ &R_S(v) = \widehat{v} && \text{on } \partial S. \end{aligned} \tag{5.28}$$

The global operator R is defined by

$$R(v)|_T = R_S(v)|_T \qquad \forall v \in U_h,\ T \in T_h.$$

The oversampled MsFEM for (2.1) is to seek $u_h \in U_h$ such that

$$\sum_{T \in T_h} \big(a_\epsilon \nabla R_{S(T)}(u_h), \nabla R_{S(T)}(v)\big)_T = \sum_{T \in T_h} \big(f, R_{S(T)}(v)\big)_T \qquad \forall v \in U_h. \tag{5.29}$$

A stability analysis for (5.29) can be done in the same fashion as for (3.2). Furthermore, Strang's second lemma can be proven as for Lemma 5.1.

Lemma 5.6. *Let u^ϵ and u_h be the respective solutions of* (2.3) *and* (5.29). *Then there is a constant $C > 0$, independent of h and ϵ, such that*

$$|u^\epsilon - R(u_h)|_{1,h} \le C \Bigg\{ \inf_{v \in U_h} |u^\epsilon - R(v)|_{1,h} + \sup_{w \in U_h, w \ne 0} \frac{\big|\sum_{T \in T_h} \big[(a_\epsilon \nabla u^\epsilon, \nabla R_S(w))_T - (f, R_S(w))_T\big]\big|}{|w|_{1,h}} \Bigg\},$$

where $S = S(T)$, $T \in T_h$.

By using this lemma and an analogous proof as for Theorem 5.5 (also see Theorem 2.15 in [7]), we can show the next theorem. Here we consider only the case $\epsilon \ll h$.

Theorem 5.7. *Assume that $U_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$. If $u^\epsilon \in U$ and $u_h \in U_h$ are the respective solutions of* (2.3) *and* (5.29), *then the error estimates hold:*

$$|u^\epsilon - R(u_h)|_{1,h} \le C \left\{ (h + \epsilon)\|f\|_{0,\Omega} + \left(\sqrt{\epsilon} + \frac{\epsilon}{h}\right) |U_0|_{1,\infty,\Omega} \right\},$$
$$\|u^\epsilon - R(u_h)\|_{0,\Omega} \le C \left\{ (h^2 + \epsilon)\|f\|_{0,\Omega} + \left(\sqrt{\epsilon} + \frac{\epsilon}{h}\right) |U_0|_{1,\infty,\Omega} \right\}.$$

We remark that, while these estimates improve those in (5.15), resonance persists.

6. A random homogenization problem. In the previous sections we have assumed that the coefficient a_ϵ in problem (2.1) has the form $a(x/\epsilon)$ or $a(x, x/\epsilon)$ and $a(x, y)$ is periodic in y. In many problems such as in porous media flows [8], this coefficient is often random. In this section we indicate how to extend the multiscale finite element analysis performed for (2.1) to a multiscale problem with a random coefficient.

Let (D, F, P) be a probability space and $a(y, \omega) = (a_{ij}(y, \omega))$ be a random field, $y \in \mathbb{R}^d$, $\omega \in D$, whose statistics is invariant under integer shifts. Furthermore, let a satisfy the uniform ellipticity condition (2.2); i.e.,

$$a_* |\zeta|^2 \le \sum_{i,j=1}^{d} a_{ij}(y, \omega) \zeta_i \zeta_j \le a^* |\zeta|^2 \qquad \forall \omega \in D,\ y, \zeta \in \mathbb{R}^d, \tag{6.1}$$

for some positive constants a_* and a^*. Problem (2.1) now takes the form

$$
\begin{aligned}
-\nabla \cdot \big(a(x/\epsilon, \omega)\nabla u^\epsilon\big) &= f \qquad \text{in } \Omega, \\
u^\epsilon &= 0 \qquad \text{on } \Gamma.
\end{aligned} \tag{6.2}
$$

As in (4.2), let χ^j satisfy [20]

$$
-\nabla_y \cdot \big(a(y, \omega)\nabla_y \chi^j\big) = \sum_{i=1}^{d} \frac{\partial}{\partial y_i} a_{ij}(y, \omega), \tag{6.3}
$$

and $\nabla\chi^j$ is assumed to be stationary under integer shifts. χ^j is generally not stationary. Define the average operator with respect to the measure P (mathematical expectation):

$$
\langle v \rangle = \mathbb{E} \int_{[0,1]^d} v(y)\, dy.
$$

The homogenized coefficient A is given by

$$
A = \langle a(\mathcal{I} + \nabla\chi) \rangle\,, \tag{6.4}
$$

where $\mathcal{I}$ is the identity matrix and $\chi = (\chi^1, \chi^2, \dots, \chi^d)^T$. With this coefficient, the variational formulation of the homogenized problem is defined as in (4.1).

For the convergence analysis in the random case, we will use an important mixing condition [17]. For a subdomain $B \subset \mathbb{R}^d$, denote by $\Phi(B)$ the σ-algebra generated by the parameters $\{a(y, \omega) : y \in B\}$. Let ζ_1 and ζ_2 be two random variables that are measurable with respect to $\Phi(B_1)$ and $\Phi(B_2)$, respectively. We assume that

$$
|\mathbb{E}(\zeta_1\zeta_2) - \mathbb{E}(\zeta_1)\mathbb{E}(\zeta_2)| \le e^{-C\,\mathrm{dist}(B_1,B_2)} \sqrt{\mathbb{E}\zeta_1^2}\sqrt{\mathbb{E}\zeta_2^2}. \tag{6.5}
$$

This type of exponential decay condition is often used for geostatistical models.

Note that the definition of the MsFEM (3.2) does not utilize any periodicity or macroscopic model. Thus, in the random case it can be defined in the same manner as in the periodic case; that is, (3.1) and (3.2) remain the same.

We now obtain error estimates for the case $\epsilon \ll h$. For the conforming multiscale finite element method, error estimates were obtained between the multiscale finite element solution and the homogenized solution U_0 [10, 14]. For the present nonconforming method, we derive similar estimates. For this, we write Strang's second lemma (Lemma 5.1) in terms of U_0.

Lemma 6.1. *Let u_h and U_0 be the respective solutions of* (3.2) *and* (4.1)*, with A given by* (6.4)*. Then there is a constant $C > 0$, independent of h and ϵ, such that*

$$
\begin{aligned}
|U_0 - R(u_h)|_{1,h} \le C\Bigg\{ &\inf_{v \in U_h} |U_0 - R(v)|_{1,h} \\
&+ \sup_{w \in U_h, w \ne 0} \frac{\left|\sum_{T \in T_h} \left[\big(a_\epsilon \nabla U_0, \nabla R_T(w)\big)_T - (f, R_T(w))_T\right]\right|}{|w|_{1,h}} \Bigg\}.
\end{aligned} \tag{6.6}
$$

Theorem 6.2. *Let u_h and U_0 be the respective solutions of* (3.2) *and* (4.1)*, where the homogenized coefficient A is now given by* (6.4)*, and $U_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$.*

Then, under condition (6.5), *we have*

$$\mathbb{E}\,|U_0 - R(u_h)|_{1,h} \le C\left(h + \left(\frac{\epsilon}{h}\right)^{\kappa}\right), \qquad (6.7)$$
$$\mathbb{E}\,\|U_0 - R(u_h)\|_{0,\Omega} \le C\left(h^2 + \left(\frac{\epsilon}{h}\right)^{\kappa}\right),$$

where

$$\kappa = \begin{cases} \dfrac{6-12\lambda}{25-8\lambda} & \textit{if } d = 3, \\ \dfrac{1}{2} & \textit{if } d = 1, \end{cases}$$

for any $0 < \lambda < 1/2$.

Proof. Again, since the conforming finite element space V_h is a subspace of U_h, it follows from Theorem 7.6 in [10] that the approximation error in (6.6) can be estimated by

$$\mathbb{E} \inf_{v\in U_h} |U_0 - R(v)|_{1,h} \le C\left\{h + \left(\frac{\epsilon}{h}\right)^{\kappa}\right\}, \qquad (6.8)$$

so it is sufficient to estimate the consistency error in (6.6).

We write

$$\begin{aligned}(6.9)\quad &\sum_{T\in T_h} \left[\left(a_\epsilon \nabla U_0, \nabla R_T(w)\right)_T - (f, R_T(w))_T\right] \\ &= \sum_{T\in T_h} \left((a_\epsilon - A)\nabla U_0, \nabla R_T(w)\right)_T + \sum_{T\in T_h} \left[\left(A\nabla U_0, \nabla R_T(w)\right)_T - (f, R_T(w))_T\right].\end{aligned}$$

Let P_h denote the standard Lagrange interpolation operator from U into U_h. Note that

$$\begin{aligned}&\sum_{T\in T_h} \left((a_\epsilon - A)\nabla U_0, \nabla R_T(w)\right)_T \\ &\quad \sum_{T\in T_h} \left((a_\epsilon - A)\nabla [U_0 - P_h U_0], \nabla R_T(w)\right)_T + \sum_{T\in T_h} \left((a_\epsilon - A)\nabla P_h U_0, \nabla R_T(w)\right)_T.\end{aligned}$$

By applying an approximation property of P_h, the first term in the right-hand side of this equation can be estimated as

$$\left|\sum_{T\in T_h} \left((a_\epsilon - A)\nabla [U_0 - P_h U_0], \nabla R_T(w)\right)_T\right| \le Ch|U_0|_{2,\Omega}|w|_{1,h}. \qquad (6.10)$$

The second term involves the convergence of a_ϵ to the homogenized matrix A. In exactly the same argument as in the conforming finite element method (see Proposition 7.8 in [10] or Theorem 1.3 in [14]), under condition (6.5), we have

$$\left|\sum_{T\in T_h} \left((a_\epsilon - A)\nabla P_h U_0, \nabla R_T(w)\right)_T\right| \le C\left(\frac{\epsilon}{h}\right)^{\kappa} |U_0|_{1,\Omega}|w|_{1,h}. \qquad (6.11)$$

Next, by using Green's formula and (4.1), for $w \in U_h$ we see that

$$\begin{aligned}\sum_{T\in T_h} & \left[\left(A\nabla U_0, \nabla R_T(w)\right)_T - (f, R_T(w))_T\right] \\ &= \sum_{T\in T_h} \left[\left(A\nabla U_0 \cdot \nu, R_T(w)\right)_{\partial T} - \left(\nabla \cdot (A\nabla U_0), R_T(w)\right)_T - (f, R_T(w))_T\right] \\ &= \sum_{T\in T_h} \left(A\nabla U_0 \cdot \nu, R_T(w)\right)_{\partial T}.\end{aligned}$$

Consequently, by the definition of $R_T(w)$ in (3.1) (i.e., $R_T(w) = w$ on ∂T),

$$\sum_{T\in T_h} \left[\left(A\nabla U_0, \nabla R_T(w)\right)_T - (f, R_T(w))_T\right] = \sum_{T\in T_h} \left(A\nabla U_0 \cdot \nu, w\right)_{\partial T}.$$

Hence, application of the standard convergence argument for the nonconforming finite element method [6, 11] implies that

$$\left|\sum_{T\in T_h} [(A\nabla U_0, \nabla R_T(w))_T - (f, R_T(w))_T]\right| \le Ch|U_0|_{2,\Omega}|w|_{1,h}. \tag{6.12}$$

Finally, applying (6.8)–(6.12) in (6.6) generates the first result in (6.7). The second result in (6.7) follows from a standard duality argument for the nonconforming finite element method [6, 11]. ∎

While estimates are given only in terms of $U_0 - R(u_h)$, they can also be shown for the error $u^\epsilon - R(u_h)$ (see section 7.3).

7. A nonlinear problem. In this section we extend the MsFEM in the nonconforming finite element setting discussed in the earlier sections to the nonlinear problem

$$\begin{aligned} -\nabla \cdot \left(a_\epsilon \nabla u^\epsilon\right) &= f \qquad \text{in } \Omega, \\ u^\epsilon &= 0 \qquad \text{on } \Gamma, \end{aligned} \tag{7.1}$$

where $a_\epsilon = a_\epsilon(x, x/\epsilon, u^\epsilon)$ now depends on the solution u^ϵ. We assume that the coefficient $a_\epsilon(x, y, z)$ is equicontinuous in z uniformly with respect to x and y and periodic in y with period $I = [0,1]^d$. Furthermore, it satisfies inequality (2.2). Under such assumptions, the solution u^ϵ converges weakly in $U = W_0^{1,p}(\Omega)$ $(p > 1)$ to the solution of the homogenized equation [5]

$$\begin{aligned} -\nabla \cdot \left(A(x, U_0)\nabla U_0\right) &= f \qquad \text{in } \Omega, \\ U_0 &= 0 \qquad \text{on } \Gamma, \end{aligned} \tag{7.2}$$

where the homogenized matrix $A = (A_{ij})$ is

$$A_{ij}(x, q) = \frac{1}{|I|}\int_I \left(a_{ij}(x, y, q) + \sum_{k=1}^d \left(a_{ik}\frac{\partial \chi^j}{\partial y_k}\right)(x, y, q)\right) dy \qquad \forall q \in \mathbb{R}$$

and χ^j satisfies, with a periodic boundary condition in y,

$$\begin{aligned} -\nabla_y \cdot \left(a_\epsilon(x, y, q)\nabla_y \chi^j\right) &= \sum_{i=1}^d \frac{\partial}{\partial y_i} a_{ij}(x, y, q), \qquad && y \in I, \\ \int_I \chi^j(x, y, q)\, dy &= 0, && q \in \mathbb{R}. \end{aligned} \tag{7.3}$$

As in (4.1), the variational form of (7.2) reads as follows: Find $U_0 \in U$ such that

$$(A(x, U_0)\nabla U_0, \nabla v) = (f, v) \qquad \forall v \in U. \tag{7.4}$$

Let $U_h \subset L^2(\Omega)$ be the nonconforming finite element space defined in the third section. For any $v \in U_h$, we define its local solution $R_T(v) \in H^1(T)$, $T \in T_h$, by

$$\begin{aligned} &(a_\epsilon(x, x/\epsilon, v)\nabla R_T(v), \nabla w)_T = 0 \qquad &&\forall w \in H_0^1(T),\\ &R_T(v) = v &&\text{on } \partial T. \end{aligned} \tag{7.5}$$

Now the MsFEM for (7.1) is the following: Find $u_h \in U_h$ such that

$$\sum_{T \in T_h} \big(a_\epsilon(x, x/\epsilon, u_h)\nabla R(u_h), \nabla R(v)\big)_T = (f, v) \qquad \forall v \in U_h. \tag{7.6}$$

Note that the local problem (7.5) is linear.

Whenever bilinear (quadratic) forms and norms involving partial derivatives are evaluated on the nonconforming finite element space U_h, they are understood in the piecewise sense, as in the definition of the norm $|\cdot|_{1,h}$. Introduce the linearized differential operator at U_0,

$$L(U_0)v = -\nabla \cdot (A(x, U_0)\nabla v + v\ A_p(x, U_0)\nabla U_0), \qquad v \in H^1(\Omega),$$

and the corresponding bilinear form

$$\hat{A}(U_0; v, w) = (A(x, U_0)\nabla v, \nabla w) + (vA_p(x, U_0)\nabla U_0, \nabla w) \qquad \forall v,\ w \in H^1(\Omega),$$

where $A_p(x, u) = \nabla_u A(x, u)$. We assume that this linearized operator is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, so U_0 is an isolated solution of (7.2), and there is $h_0 > 0$ such that, for $0 < h < h_0$ [6, 22],

$$\sup_{w \in U_h} \frac{\hat{A}(U_0; v, w)}{\|w\|_{1,\Omega}} \ge C_0\|v\|_{1,\Omega} \qquad \forall v \in U_h, \tag{7.7}$$

where $C_0 > 0$ is independent of h.

For any $v, v_1, w \in U$, we define

$$\mathcal{R}(v, v_1, w) = A(v_1, w) - A(v, w) - \hat{A}(v; v_1 - v, w).$$

If $v, v_1 \in U$ satisfy $\|v\|_{1,\infty,\Omega} + \|v_1\|_{1,\infty,\Omega} \le M$, then it follows from [24] that

$$|\mathcal{R}(v, v_1, w)| \le C(M)\left(\|e\|_{0,2p}^2 + \|e\nabla e\|_{0,p}\right)\|\nabla w\|_{0,q}, \qquad e = v - v_1,\ \frac{1}{p} + \frac{1}{q} = 1. \tag{7.8}$$

It follows from the definition of $\mathcal{R}$ and (7.4) that $u_h \in U_h$ is the solution of (7.6) if and only if

$$\begin{aligned} \hat{A}(U_0; U_0 - u_h, w) = \mathcal{R}(U_0, u_h, w) &+ [A(U_0, w) - (f, w)]\\ &+ [A_h(u_h, w) - A(u_h, w)] \qquad \forall w \in U_h, \end{aligned} \tag{7.9}$$

where $A_h(u_h, w) = \sum_{T \in T_h}(a_\epsilon(x, x/\epsilon, u_h)\nabla R(u_h), \nabla R(w))_T$. Define

$$E(v, w) = A_h(v, w) - A(v, w) \qquad \forall v,\ w \in U_h,$$

and

$$\overline{E} = \max_{v \in U_h \cap W^{1,\infty}(\Omega), w \in U_h} \frac{|E(v, w)|}{\|v\|_{1,\Omega}\|w\|_{1,\Omega}}.$$

7.1. Existence and uniqueness of a solution. To prove the existence and uniqueness of a solution to (7.6), we introduce the projection of U_0 into U_h through the linearized bilinear form $\hat{A}$:

$$\hat{A}(U_0; P_h U_0, v) = \hat{A}(U_0; U_0, v) \qquad \forall v \in U_h. \tag{7.10}$$

It follows from (7.7) that $P_h U_0$ exists, is unique for $0 < h < h_0$, and satisfies [13]

$$\|U_0 - P_h U_0\|_{1,\infty,\Omega} \leq Ch, \quad \|U_0 - P_h U_0\|_{1,\Omega} \leq Ch \tag{7.11}$$

if $U_0 \in W^{2,\infty}(\Omega)$. When $U_0 \in W^{2,p}(\Omega)$ $(p > d)$, it holds that

$$\|U_0 - P_h U_0\|_{1,\infty,\Omega} \leq Ch^{1-d/p}. \tag{7.12}$$

Finally, for a given $x \in \Omega$, we define the discrete Green's function $G_h^x \in U_h$ by

$$\hat{A}(U_0; v, G_h^x) = \partial v(x) \qquad \forall v \in U_h, \tag{7.13}$$

where ∂v indicates any of the partial derivatives $\partial v/\partial x_i$ $(i = 1, 2, \ldots, d)$. This function satisfies

$$\|G_h^x\|_{1,1,\Omega} \leq C|\ln h|. \tag{7.14}$$

THEOREM 7.1. *Assume that L is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ and $U_0 \in U \cap W^{2,p}(\Omega)$, with $p > d$. In addition, assume that there are constants C_1 and h_0 such that, for $0 < h \leq h_0$,*

$$\overline{E}^{1/2}|\ln h| \leq C_1. \tag{7.15}$$

Then problem (7.6) *has a solution u_h satisfying*

$$\begin{aligned} &\|u_h - P_h U_0\|_{1,\infty,\Omega} \leq \overline{E}^{1/2} + h^{1-d/p}, \\ &\|u_h - U_0\|_{1,\infty,\Omega} \leq C\left(\overline{E}^{1/2} + h^{1-d/p}\right). \end{aligned} \tag{7.16}$$

Furthermore, if, for all $v_1, v_2 \in U_h \cap W^{1,\infty}(\Omega)$ and $w \in U_h$, with $\|v_1\|_{1,\infty,\Omega} + \|v_2\|_{1,\infty,\Omega} \leq M$, there is a constant $\zeta_0(M)$, with $0 < \zeta_0/C_0 < 1$, such that

$$|E(v_1, w) - E(v_2, w)| \leq \zeta_0 \|v_1 - v_2\|_{1,\Omega} \|w\|_{1,\Omega}, \tag{7.17}$$

then this solution u_h is locally unique, where C_0 is given by (7.7).

Proof. We define the nonlinear mapping $\mathcal{L} : U_h \to U_h$:

$$\hat{A}(U_0; \mathcal{L}(v), w) = \hat{A}(U_0; U_0, w) - \mathcal{R}(U_0, v, w) + A(v, w) - A_h(v, w) \qquad \forall w \in U_h.$$

This mapping is continuous by using (7.7) and (7.8). We also define the set

$$B = \left\{v \in U_h : \|v - P_h U_0\|_{1,\infty,\Omega} \leq \overline{E}^{1/2} + h^{1-d/p}\right\}.$$

Note that, by (7.10),

$$\hat{A}(U_0; \mathcal{L}(v) - P_h U_0, w) = -\mathcal{R}(U_0, v, w) + A(v, w) - A_h(v, w) \qquad \forall w \in U_h.$$

By choosing $w = G_h^x$ in this equation and applying (7.8), (7.12), (7.14), and (7.15), we see that, with $v \in B$ and $\|v\|_{1,\infty,\Omega} \leq M$,

$$\begin{aligned}\|\mathcal{L}(v) - P_h U_0\|_{1,\infty,\Omega} &\leq C(M)\left(\|U_0 - v\|^2_{1,\infty,\Omega} + \overline{E}\right)|\ln h| \\ &\leq C(M)\left(\|U_0 - P_h U_0\|^2_{1,\infty,\Omega} + \|P_h U_0 - v\|^2_{1,\infty,\Omega} + \overline{E}\right)|\ln h| \\ &\leq C(M)\left(h^{2-2d/p} + \overline{E}\right)|\ln h| \\ &\leq \overline{E}^{1/2} + h^{1-d/p},\end{aligned}$$

which implies that $\mathcal{L}(B) \subset B$ with an appropriate choice of C_1. The Brouwer fixed point theorem means that there is a $u_h \in B$ such that $\mathcal{L}(u_h) = u_h$.

To prove the uniqueness, let u_h^1 and u_h^2 be two solutions of (7.6). Then it follows from (7.7) that, with $u_h^t = (1-t)u_h^2 + tu_h^1$,

$$\begin{aligned}C_0\|u_h^1 - u_h^2\|_{1,\Omega} &\leq \sup_{w \in U_h} \frac{\int_0^1 \hat{A}(u_h^t; u_h^1 - u_h^2, w)\, dt}{\|w\|_{1,\Omega}} \\ &\leq \sup_{w \in U_h} \frac{|A(u_h^1, w) - A(u_h^2, w)|}{\|w\|_{1,\Omega}}.\end{aligned}$$

Note that, by (7.7),

$$A(u_h^1, w) - A(u_h^2, w) = \left[A(u_h^1, w) - A_h(u_h^1, w)\right] - \left[A(u_h^2, w) - A_h(u_h^2, w)\right].$$

Consequently, by using (7.17), we see that

$$\|u_h^1 - u_h^2\|_{1,\Omega} \leq \frac{\zeta_0}{C_0}\|u_h^1 - u_h^2\|_{1,\Omega}.$$

Since $\zeta_0/C_0 < 1$ via assumption, $u_h^1 = u_h^2$. Therefore, the solution u_h is locally unique. □

7.2. Error estimates. The multiscale finite element solution u_h in the next theorem refers to the one that satisfies the conditions in Theorem 7.1. In the previous sections both of the cases $h < \epsilon$ and $h > \epsilon$ were analyzed for convergence of the linear MsFEM. Here we focus on the case $\epsilon < h$, where $\epsilon > 0$ is assumed sufficiently small.

THEOREM 7.2. *Let U_0 and u_h be the solutions of* (7.4) *and* (7.6)*, respectively, and $U_0 \in W^{2,\infty}(\Omega)$. Then there is $h_0 > 0$ such that, for $0 < h < h_0$,*

$$\|U_0 - u_h\|_{1,\Omega} \leq C\left(h + \sqrt{\frac{\epsilon}{h}}\right), \quad \|U_0 - u_h\|_{1,\infty,\Omega} \leq C\left(h + \sqrt{\frac{\epsilon}{h}}\right)|\ln h|, \tag{7.18}$$

provided that ϵ/h is sufficiently small.

Proof. By taking $w = P_h U_0 - u_h$ in (7.9) and using (7.7), (7.8), and a similar argument as for (6.12), we see that

$$\|P_h U_0 - u_h\|_{1,\Omega} \leq C\left(\|U_0 - u_h\|^2_{1,4,\Omega} + \overline{E} + h\right). \tag{7.19}$$

As in the conforming case (see Proposition 4.3 in [10]), it can be shown that

$$\overline{E} \leq C\sqrt{\frac{\epsilon}{h}}, \tag{7.20}$$

provided that h and ϵ/h are sufficiently small. Also, by applying an interpolation inequality, we have

$$\|U_0 - u_h\|^2_{1,4,\Omega} \le \|U_0 - u_h\|_{1,\Omega}\|U_0 - u_h\|_{1,\infty,\Omega}. \tag{7.21}$$

Consequently, combining (7.19)–(7.21) and using (7.11) and Theorem 7.1 yields the first result in (7.18).

By choosing $w = G_h^x$ in (7.9) and using (7.8), (7.11), and (7.14), we see that

$$\|P_hU_0 - u_h\|_{1,\infty,\Omega} \le C\overline{E}|\ln h|\|P_hU_0 - u_h\|_{1,\infty,\Omega} + C\left(\|U_0 - u_h\|^2_{1,\infty,\Omega} + \overline{E} + h\right)|\ln h|.$$

By applying (7.20), if ϵ/h is sufficiently small, it follows that

$$\|P_hU_0 - u_h\|_{1,\infty,\Omega} \le C\left(\|U_0 - u_h\|^2_{1,\infty,\Omega} + \overline{E} + h\right)|\ln h|.$$

As a result, by applying Theorem 7.1, we obtain

$$\|U_0 - u_h\|_{1,\infty,\Omega} \le C\left(\|P_hU_0 - U_0\|_{1,\infty,\Omega} + \left(\overline{E} + h\right)|\ln h|\right),$$

which, together with (7.11), implies the second inequality in (7.18). ∎

7.3. Approximation to u^ϵ. Theorem 7.2 shows that the multiscale finite element solution u_h of (7.6) is a good approximation of the macroscopic solution U_0. We now consider an approximation to the solution u^ϵ of (7.1).

For each $T \in T_h$, let x_T be the barycenter of T and u_h be the solution of (7.6). Note that

$$\nabla R_T(u_h) = \nabla u_h + \sum_{k=1}^{d} \nabla_y \chi^k\left(x_T, \frac{x}{\epsilon}, u_h(x_T)\right)\frac{\partial u_h}{\partial x_k}, \qquad T \in T_h. \tag{7.22}$$

Define the first-order approximation of u_1^ϵ by

$$u_1^\epsilon = U_0 + \epsilon\sum_{k=1}^{d}\chi^k\left(x, \frac{x}{\epsilon}, U_0(x)\right)\frac{\partial U_0}{\partial x_k}.$$

Clearly,

$$\nabla u_1^\epsilon = \nabla U_0 + \sum_{k=1}^{d}\left\{\left(\epsilon\nabla\chi^k + \nabla_y\chi^k\right)\frac{\partial U_0}{\partial x_k} + \epsilon\frac{\partial\chi^k}{\partial U_0}\frac{\partial U_0}{\partial x_k}\nabla U_0 + \epsilon\chi^k\nabla\frac{\partial U_0}{\partial x_k}\right\}. \tag{7.23}$$

Combining (7.22) and (7.23) gives

(7.24)

$$\begin{aligned}
|u_1^\epsilon - R(u_h)|_{1,T} &\le C\left(|U_0 - u_h|_{1,T} + \epsilon\left[\|U_0\|_{2,T} + \|U_0\|^2_{1,4,T}\right]\right)\\
&\quad + \left\|\sum_{k=1}^{d}\left[\nabla_y\chi^k\left(x_T, \frac{x}{\epsilon}, u_h(x_T)\right) - \nabla_y\chi^k\left(x, \frac{x}{\epsilon}, U_0(x)\right)\right]\frac{\partial U_0}{\partial x_k}\right\|_{0,T}\\
&\le C\Big\{|U_0 - u_h|_{1,T} + |U_0 - u_h|_{1,\infty,T}|U_0|_{1,T}\\
&\quad + (\epsilon + h)\left(\|U_0\|_{2,T} + \|U_0\|^2_{1,4,T} + |U_0|_{1,\infty,T}|U_0|_{1,T}\right)\Big\}.
\end{aligned}$$

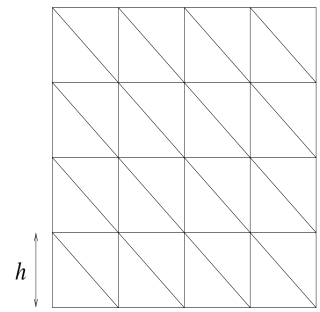

FIG. 1. *A triangulation of a unit square.*

A classical estimate for $u^\epsilon - u_1^\epsilon$ [4, 5, 25] gives

$$\|u^\epsilon - u_1^\epsilon\|_{1,\Omega} \le C\sqrt{\epsilon}. \tag{7.25}$$

Finally, by combining (7.18), (7.24), and (7.25), we obtain the next theorem.

THEOREM 7.3. *Let u^ϵ be the solution of* (7.1)*, $R(u_h)$ be defined by* (7.22)*, and $U_0 \in W^{2,\infty}(\Omega)$. Then there is $h_0 > 0$ such that, for $0 < h < h_0$,*

$$|u^\epsilon - R(u_h)|_{1,h} \le C\left(h + \sqrt{\epsilon} + \sqrt{\frac{\epsilon}{h}}\right)|\ln h|, \tag{7.26}$$

provided that ϵ/h is sufficiently small.

8. Numerical experiments. Numerical experiments are not available for the MsFEM defined in the equivalent formulation (3.2) even in the conforming case. Before numerical experiments are presented for the nonconforming MsFEM, we first give them for the conforming MsFEM in formulation (3.2). The focus here is to check the accuracy theory established in the previous sections. More numerical results will be reported in our future research.

8.1. Conforming finite elements. The domain Ω is assumed to be a unit square, which is triangulated into $2(m+1)^2$ triangles as shown in Figure 1. To solve the local problem (3.1), each triangle is divided into $(n+1)^2$ smaller equal triangles. The traditional conforming P_1 finite element method directly used to solve problem (2.1) on the fine grid requires memory of order $O(n^2m^2)$, while MsFEM's memory is of order $O(n^2+m^2)$, so there is a significant reduction in memory. We are interested in incorporating as much fine-scale information into the solution as possible, and thus m is usually much larger then n.

Example 1. The first example exploits a rapidly varying coefficient a_ϵ:

$$a_\epsilon = \frac{1}{2 + 1.5\sin(2\pi(x_1+x_2)/\epsilon)},$$

which is plotted in Figure 2 for $\epsilon = 0.08$. The right-hand side function f is chosen in such a way that an analytic solution results. As an example, for $u = x_1^2x_2^2 - x_1^2x_2 - x_1x_2^2 + x_1x_2$ and $n = 8$, the convergence results in the H^1, L^2, and L^∞ norms are given in Table 1 for the MsFEM. These estimates are in good agreement with our theory; the numerical H^1 estimates are better than theoretical ones.

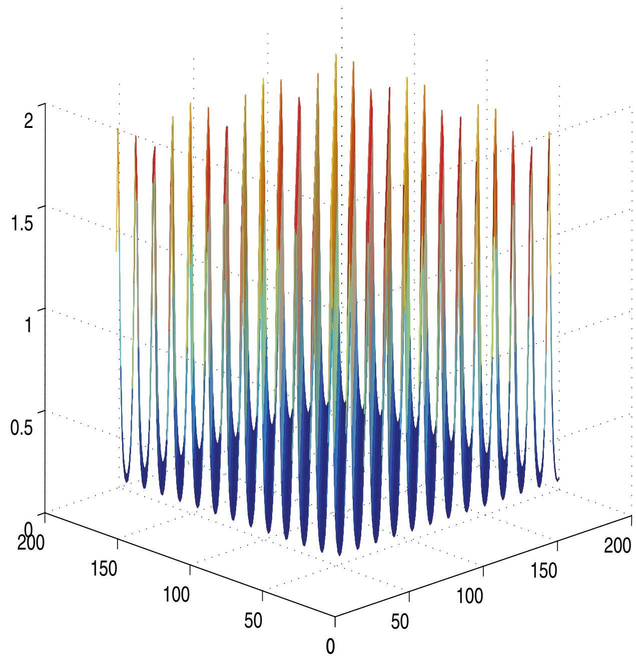

Fig. 2. *The plot of a_ϵ in Example 1 with $\epsilon = 0.08$.*

Table 1
Convergence results for Example 1 with $\epsilon = 0.08$ and $n = 8$.

m	H^1	Rate	L^2	Rate	L^∞	Rate
9	2.8571e-04	-	3.1162e-04	-	5.1649e-4	-
19	8.3350e-05	1.78	1.0640e-04	1.55	2.7913e-4	0.89
39	2.4409e-05	1.77	2.8078e-05	1.92	6.4062e-5	2.12
79	6.7166e-06	1.86	7.1565e-06	1.97	1.4142e-5	2.18

Table 2
Convergence results for Example 2 with $\epsilon = 1/60$ and $n = 8$.

m	H^1	Rate	L^2	Rate	L^∞	Rate
9	2.1028e-2	-	2.4252e-2	-	4.6161e-2	-
19	5.2847e-3	1.99	5.6493e-3	2.10	1.3671e-2	1.76
39	1.3644e-3	1.95	1.4472e-3	1.96	3.5562e-3	1.94
79	3.4496e-4	1.98	3.5427e-4	2.03	9.0519e-4	1.97

Example 2. To test our method further, we consider a different choice of the rapidly oscillating coefficient:

$$a_\epsilon = \frac{1.5 + \sin(2\pi x_1/\epsilon)}{1.5 + \sin(2\pi x_2/\epsilon)} + \frac{1.5 + \sin(2\pi x_2/\epsilon)}{1.5 + \sin(2\pi x_1/\epsilon)} + \sin(4x_1^2 x_2^2) + 1.$$

For this example, the analytic solution is $u = \sin(2\pi x_1)\sin(2\pi x_2)$, and the convergence results are presented in Table 2. Similar results as those in Example 1 are obtained; here the numerical H^1 estimates are of order two, which is superconvergence.

TABLE 3
Convergence results for Example 3 with $\epsilon = 0.08$ and $n = 8$.

	Traditional FEM		MsFEM	
m	L^2	Rate	L^2	Rate
9	1.1760+00	-	4.1913e-2	-
19	2.6424+00	-1.16	1.0698e-2	1.97
39	1.2723+00	1.05	2.8077e-3	1.93
79	2.3862e-1	2.41	8.1074e-4	1.79

TABLE 4
Convergence results for the nonconforming MsFEM with $\epsilon = 0.08$ and $n = 8$.

m	H^1	Rate	L^2	Rate	L^∞	Rate
9	3.0277e-05	-	1.1367e-04	-	0.0057	-
19	6.5849e-06	2.20	1.2809e-05	3.14	0.0015	1.93
39	1.5790e-06	2.06	1.5208e-06	3.07	3.6452e-04	2.04
79	3.8324e-07	2.04	1.8527e-07	3.03	9.1115e-05	2.00

Example 3. In the third example, we compare the traditional finite element method and MsFEM for numerically solving (2.1) with

$$a_\epsilon = \frac{1}{(2 + 1.8\sin(2\pi x_1/\epsilon))(2 + 1.8\sin(2\pi x_2/\epsilon))}.$$

For the analytic solution $u = \cos(2\pi x_1)\cos(2\pi x_2)$, the comparison is given in Table 3, which indicates the superiority of MsFEM.

8.2. Nonconforming finite elements. We now present numerical experiments for the nonconforming multiscale finite element method (3.2).

Example 4. As an example, we only consider problem (2.1) with the rapidly oscillating coefficient:

$$a_\epsilon = \frac{1}{2 + \sin((x_1 + x_2)/\epsilon)}.$$

For the analytic solution $u = \sin(2\pi x_1)\sin(x_2)$, the convergence results in the H^1 (discrete), L^2, and L^∞ norms are given in Table 4 for the MsFEM. For this particular example, superconvergence is observed for both the H^1 and L^2 errors.

Example 5. Finally, we test the MsFEM with the oversampling technique as presented in section 5.3, where problem (2.1) has the following data:

$$\begin{aligned} a_\epsilon &= \frac{1 + 0.9\sin(2\pi x_1/\epsilon)}{1 + 0.9\sin(2\pi x_2/\epsilon)} + \frac{1 + 0.9\sin(2\pi x_2/\epsilon)}{1 + 0.9\sin(2\pi x_1/\epsilon)}, \\ f &= -1. \end{aligned}$$

The exact solution of this test example is unknown, so the coarse mesh solutions obtained by the oversampled nonconforming MsFEM are compared with the solution u_h obtained by using the standard conforming method over the mesh $nm = 1264$. For a fixed $\epsilon/h = 0.5$, the error estimates in the L^2 norm are given in Table 5, which shows a first-order convergence.

TABLE 5
Convergence results for the oversampled nonconforming MsFEM with $\epsilon/h = 0.5$ and $n = 8$.

m	ϵ	L^2	Rate
9	1/20	2.2344e-04	-
19	1/40	1.1271e-04	0.9874
39	1/80	5.5832e-05	1.0135
79	1/160	2.7822e-05	1.0049

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