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ANALYSIS OF THE MULTISCALE FINITE ELEMENT METHOD FOR NONLINEAR AND RANDOM HOMOGENIZATION PROBLEMS*

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Abstract. In this paper we study the convergence of the multiscale finite element method for nonlinear and random homogenization problems. Error estimates similar to those for linear homogenization problems are obtained here.

Key words. multiscale problem, multiscale finite element method, finite element, nonlinear problems, random problems, oversampling technique, convergence, stability, error estimate

AMS subject classifications. 35K60, 35K65, 76S05, 76T05

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1. Introduction.

Hou and Wu [16] introduced the multiscale finite element method for numerical solution of multiscale problems that are described by partial differential equations with highly oscillatory coefficients. The main idea of this method is to incorporate the microscale information of a multiscale differential problem into finite element basis functions. It is through these modified bases and finite element formulations that the effect of microscales on macroscales can be correctly captured.

The convergence analysis of the method was given in [17] for a two-scale linear homogenization problem with periodic coefficients. It was proven that the multiscale finite element solution converges to the homogenized solution as $h,\ \epsilon \to 0$, where h is the mesh size and ϵ is the small scale in the solution. The analysis also indicated that a resonance error exists between the grid scale and the scales of the homogenization problem. This is a common feature in some numerical upscaling techniques. This error represents a mismatch between the local construction of the multiscale basis functions and the global nature of the continuous problem. An oversampling technique was analyzed in [13] to reduce the resonance error. The idea of this technique is to construct the local basis functions over a domain with size larger than h to reduce the boundary layer effect present in the first order corrector of the local solution.

A convergence analysis of the multiscale finite element method and its oversampled version for nonlinear problems was recently given by Efendiev [11] and Efendiev, Hou, and Ginting [12], and error estimates were obtained for monotone operators (see the next section). These monotone operators do not cover the nonlinear problem studied here, which is a natural extension of the usual linear homogenization problems with highly oscillatory coefficients [5, 16, 17]. The primary goal of this paper is to derive error estimates for the nonlinear problem under consideration. We show that error estimates similar to those for linear homogenization problems hold

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for this problem as well. We also indicate how to extend this error analysis to random homogenization problems.

The present analysis is based on an equivalent formulation for the multiscale finite element method recently introduced by Chen [5], which utilizes standard basis functions of finite element spaces but modifies the bilinear (quadratic) form in the finite element formulation of the underlying multiscale problems. This new formulation captures the macroscale structure of the solution of a differential multiscale problem through the modification of this bilinear form. It is a general approach that can handle a large variety of differential problems, periodic or nonperiodic, linear or nonlinear, and stationary or dynamic, as shown here. A similar idea using the operator approach was employed by Arbogast [1].

The paper is organized as follows. In the next section we present a continuous two-scale nonlinear problem, the multiscale finite element method, and existing error estimates. Existence and uniqueness of a multiscale finite element solution is shown in the third section. An improved error analysis is given in the fourth section. An oversampling technique for the nonlinear problem is presented in the fifth section, and a simple reconstruction trick to retrieve the microscopic information is described in the sixth section. Finally, an extension to a random homogenization problem is given in the seventh section. As a general remark, the generic constant $C > 0$ (with or without a subscript) is assumed to be independent of the mesh size h and the microscale ϵ throughout this paper.

2. Existing error estimates for nonlinear problems. Let Ω be a bounded domain in $\mathbb{R}^d$, $1 \leq d \leq 3$, with Lipschitz boundary Γ. For a subdomain $D \subset \Omega$, each integer $m \geq 0$, and each real number $1 \leq p \leq \infty$, $W^{m,p}(D)$ indicates the usual Sobolev space of real functions that have all their weak derivatives of order up to m in the Lebesgue space $L^p(D)$. The norm and seminorm of $W^{m,p}(D)$ are denoted by $\|\cdot\|_{m,p,D}$ and $|\cdot|_{m,p,D}$, respectively. When $p = 2$, $W^{m,p}(D)$ is written as $H^m(D)$ with the norm $\|\cdot\|_{m,D}$ and the seminorm $|\cdot|_{m,D}$. We also use the space

$$W_0^{1,p}(D) = \{v \in W^{1,p}(D) : v|_{\partial D} = 0\}, \qquad p > 1.$$

Again, when $p = 2$, it is written as $H_0^1(D)$.

We consider the nonlinear problem

$$\begin{aligned} -\nabla \cdot \big(a_\epsilon \nabla u^\epsilon\big) &= f \qquad && \text{in } \Omega, \\ u^\epsilon &= 0 && \text{on } \Gamma, \end{aligned} \tag{2.1}$$

where $a_\epsilon = a(x, x/\epsilon, u^\epsilon)$ depends on the solution u^ϵ. In problem (2.1), the multiscale feature is reflected in the oscillatory nature of the coefficient a_ϵ for $\epsilon \ll 1$, which represents the microscale.

We assume that the coefficient $a(x, y, z)$ is equicontinuous in z uniformly with respect to x and y and periodic in y with period $I = [0, 1]^d$. Furthermore, it satisfies

$$a_*|\zeta|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x, y, q)\zeta_i\zeta_j \leq a^*|\zeta|^2 \qquad \forall x \in \Omega,\ y,\ \zeta \in \mathbb{R}^d,\ q \in \mathbb{R}, \tag{2.2}$$

for some positive constants a_* and a^*. Under such assumptions, the solution u^ϵ of problem (2.1) converges weakly in $\mathcal{U} = W_0^{1,p}(\Omega)$ $(p > 1)$ to the solution of the homogenized equation [3]

$$\begin{aligned} -\nabla \cdot \big(\mathcal{A}(x, U_0)\nabla U_0\big) &= f \qquad && \text{in } \Omega, \\ U_0 &= 0 && \text{on } \Gamma, \end{aligned} \tag{2.3}$$

where the homogenized matrix $\mathcal{A} = (\mathcal{A}_{ij})$ is

$$\mathcal{A}_{ij}(x,q) = \frac{1}{|I|}\int_I \left(a_{ij}(x,y,q) + \sum_{k=1}^{d} \left(a_{ik}\frac{\partial \chi^j}{\partial y_k}\right)(x,y,q)\right) dy \qquad \forall q \in \mathbb{R},$$

and χ^j satisfies, with a periodic boundary condition in y,

$$\begin{aligned} -\nabla_y \cdot \big(a(x,y,q)\nabla_y \chi^j\big) &= \sum_{i=1}^{d} \frac{\partial}{\partial y_i} a_{ij}(x,y,q), && y \in I, \\ \int_I \chi^j(x,y,q)\, dy &= 0, && q \in \mathbb{R}. \end{aligned} \tag{2.4}$$

The variational form of (2.3) reads as follows: Find $U_0 \in \mathcal{U}$ such that

$$A(U_0, v) \equiv (\mathcal{A}(x,U_0)\nabla U_0, \nabla v) = (f,v) \qquad \forall v \in \mathcal{U}. \tag{2.5}$$

For $h > 0$, let T_h be a regular, quasi-uniform macroscale triangulation of Ω [4, 7] into simplices, where the mesh size h resolves the variations of Ω, f, and the slow variable of a_ϵ. Associated with T_h, let $\mathcal{U}_h \subset \mathcal{U}$ be the finite element space of piecewise linear functions over simplices so that for any $v \in \mathcal{U} \cap H^2(\Omega)$, there exists a $v_h \in \mathcal{U}_h$ satisfying the approximation property

$$\|v - v_h\|_{0,\Omega} + h\|v - v_h\|_{1,\Omega} \le Ch^2 |v|_{l,\Omega}. \tag{2.6}$$

For any $v \in \mathcal{U}_h$, we define its local solution $R_T(v) \in H^1(T)$, $T \in T_h$, by

$$\begin{aligned} &(a(x,x/\epsilon,v)\nabla R_T(v), \nabla w)_T = 0 && \forall w \in H_0^1(T), \\ &R_T(v) = v && \text{on } \partial T. \end{aligned} \tag{2.7}$$

The global operator R is then given by

$$R(v)|_T = R_T(v) \qquad \forall v \in \mathcal{U}_h,\ T \in T_h.$$

It is easy to see that $R(v) \in \mathcal{U}$, $v \in \mathcal{U}_h$. Note that the local problem (2.7) is linear. In the case without ambiguity in the context, the subscript T in R will be omitted.

Define

$$A_h(v,w) = \big(a(x,x/\epsilon,v)\nabla R(v), \nabla R(w)\big), \qquad v,\ w \in \mathcal{U}_h.$$

The multiscale finite element method (MsFEM) for (2.1) is as follows: Find $u_h \in \mathcal{U}_h$ such that

$$A_h(u_h, v) = (f,v) \qquad \forall v \in \mathcal{U}_h. \tag{2.8}$$

Note that the major difference between (2.8) and the standard Galerkin finite element method lies in the modification of the bilinear form, which needs the solution of local problems (2.7). It is through these local problems and the finite element formulation that the effect of microscales on macroscales can be correctly captured. Since these local problems are independent of each other, they can be solved in parallel.

For a linear counterpart of problem (2.1) where the coefficient $a_\epsilon = a(x, x/\epsilon)$ does not depend on the solution u^ϵ, the following error estimate was obtained for method (2.8) [5, 17]:

$$(2.9)\qquad \begin{aligned} |u^\epsilon - R(u_h)|_{1,\Omega} &\le C\left\{(h+\epsilon)\|f\|_{0,\Omega} + \sqrt{\frac{\epsilon}{h}}\;|U_0|_{1,\infty,\Omega}\right\},\\ \|u^\epsilon - R(u_h)\|_{0,\Omega} &\le C\left\{(h^2+\epsilon)\|f\|_{0,\Omega} + \sqrt{\frac{\epsilon}{h}}\;|U_0|_{1,\infty,\Omega}\right\}, \end{aligned}$$

provided that $U_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$.

Recently, Efendiev, Hou, and Ginting [12] studied convergence of the MsFEM for a nonlinear problem analogous to (2.1):

$$(2.10)\qquad -\nabla \cdot a(x, u^\epsilon, \nabla u^\epsilon) = f.$$

Under the assumptions that $a(x, u^\epsilon, \nabla u^\epsilon) = a(x/\epsilon, \nabla u^\epsilon)$ and

$$\begin{aligned} &|a(x,\xi)| \le C|\xi|^{p-1},\\ &(a(x,\xi_1) - a(x,\xi_2), \xi_1 - \xi_2) \ge C|\xi_1 - \xi_2|^p,\\ &(a(x,\xi),\xi) \ge C|\xi|^p,\\ &|a(x,\xi_1) - a(x,\xi_2)| \le CH(\xi_1,\xi_2,p-1-s)|\xi_1 - \xi_2|^s, \end{aligned}$$

where $s > 0$ and $p > 1$, the following error estimate was derived (see [12, Theorem 3.2]):

$$(2.11)\qquad \|u - u_h\|_{1,p,\Omega}^p \le C\left(\left(\frac{\epsilon}{h}\right)^{\frac{s}{(p-1)(p-s)}} + \left(\frac{\epsilon}{h}\right)^{\frac{p}{p-1}} + h^{\frac{p}{p-1}}\right).$$

Obviously, the assumptions made on a_ϵ exclude the nonlinear problem (2.1). The convergence result in [12] was obtained for the general nonlinear case (2.10); however, there is no explicit convergence for the case considered in this paper because of very weak assumptions made in the way the coefficients depend on u^ϵ. The aim of this paper is to obtain error estimates for the MsFEM (2.8). In particular, we will derive error estimates similar to (2.9) under much weaker assumptions on the coefficient a_ϵ in (2.1). The error analysis is inspired by E [9] and E, Ming, and Zhang [10] for the heterogeneous multiscale method.

3. Existence and uniqueness of a solution. It is known that the error analysis of the MsFEM for the case $h < \epsilon$ is different from that for the case $h > \epsilon$. In the former case, the MsFEM has error estimates similar to those for the traditional finite element method [5, 17]. It is the latter case that is of interest and is being investigated in this paper. The argument below requires that $\epsilon > 0$ be sufficiently small.

Introduce the linearized differential operator at U_0

$$L(U_0)v = -\nabla \cdot (\mathcal{A}(x, U_0)\nabla v + v\;\mathcal{A}_p(x, U_0)\nabla U_0), \qquad v \in H^1(\Omega),$$

and the corresponding bilinear form

$$\hat{A}(U_0; v, w) = (\mathcal{A}(x, U_0)\nabla v, \nabla w) + (v\mathcal{A}_p(x, U_0)\nabla U_0, \nabla w) \qquad \forall v,\; w \in H^1(\Omega),$$

where $\mathcal{A}_p(x, u) = \nabla_u \mathcal{A}(x, u)$. We assume that this linearized operator is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, and so U_0 is an isolated solution of (2.5) and there is

$h_0 > 0$ such that for $0 < h < h_0$ [19],

$$\sup_{w \in \mathcal{U}_h} \frac{\hat{A}(U_0; v, w)}{\|w\|_{1,\Omega}} \geq C_0 \|v\|_{1,\Omega} \qquad \forall v \in \mathcal{U}_h, \tag{3.1}$$

where $C_0 > 0$ is independent of h.

For any $v, v_h, w \in \mathcal{U}$, we define

$$\mathcal{R}(v, v_h, w) = A(v_h, w) - A(v, w) - \hat{A}(v; v_h - v, w).$$

If $v, v_h \in \mathcal{U}$ satisfy $\|v\|_{1,\infty,\Omega} + \|v_h\|_{1,\infty,\Omega} \leq M$, then it follows from [20] that

(3.2)

$$|\mathcal{R}(v, v_h, w)| \leq C(M) \left(\|e_h\|_{0,2p}^2 + \|e_h \nabla e_h\|_{0,p} \right) \|\nabla w\|_{0,q}, \qquad e_h = v - v_h, \ \frac{1}{p} + \frac{1}{q} = 1.$$

It follows from the definition of $\mathcal{R}$ and (2.5) that $u_h \in \mathcal{U}_h$ is the solution of (2.8) if and only if

$$\hat{A}(U_0; U_0 - u_h, w) = \mathcal{R}(U_0, u_h, w) + A_h(u_h, w) - A(u_h, w) \qquad \forall w \in \mathcal{U}_h. \tag{3.3}$$

Define

$$E(v, w) = A_h(v, w) - A(v, w) \qquad \forall v, \ w \in \mathcal{U}_h,$$

and

$$\overline{E} = \max_{v \in \mathcal{U}_h \cap W^{1,\infty}(\Omega), w \in \mathcal{U}_h} \frac{|E(v, w)|}{\|v\|_{1,\Omega} \|w\|_{1,\Omega}}.$$

The following equivalence will be used [5]:

$$C_1' |v|_{1,T} \leq |R_T(v)|_{1,T} \leq C_2' |v|_{1,T} \qquad \forall T \in T_h, \ v \in \mathcal{U}_h. \tag{3.4}$$

To prove existence and uniqueness of a solution to (2.8), we introduce the projection of U_0 into $\mathcal{U}_h$ through the linearized bilinear form $\hat{A}$:

$$\hat{A}(U_0; P_h U_0, v) = \hat{A}(U_0; U_0, v) \qquad \forall v \in \mathcal{U}_h. \tag{3.5}$$

It follows from (3.1) that $P_h U_0$ exists and is unique for $0 < h < h_0$, and it satisfies [8]

$$\|U_0 - P_h U_0\|_{1,\infty,\Omega} \leq Ch, \quad \|U_0 - P_h U_0\|_{1,\Omega} \leq Ch \tag{3.6}$$

if $U_0 \in W^{2,\infty}(\Omega)$. When $U_0 \in W^{2,p}(\Omega)$ $(p > d)$, it holds that

$$\|U_0 - P_h U_0\|_{1,\infty,\Omega} \leq Ch^{1-d/p}. \tag{3.7}$$

Finally, for a given $x \in \Omega$, we define the discrete Green function $G_h^x \in \mathcal{U}_h$ by

$$\hat{A}(U_0; v, G_h^x) = \partial v(x) \qquad \forall v \in \mathcal{U}_h, \tag{3.8}$$

where ∂v indicates any of the partial derivatives $\partial v / \partial x_i$ $(i = 1, 2, \ldots, d)$. This function satisfies

$$\|G_h^x\|_{1,1,\Omega} \leq C |\ln h|. \tag{3.9}$$

THEOREM 3.1. *Assume that L is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ and $U_0 \in \mathcal{U} \cap W^{2,p}(\Omega)$ with $p > d$. In addition, assume that $\overline{E}$ is bounded and there are constants C_1 and h_1 such that for $0 < h \le h_1$,*

$$\overline{E}^{1/2} |\ln h| \le C_1. \tag{3.10}$$

Then problem (2.8) has a solution u_h satisfying

$$\begin{aligned} \|u_h - P_h U_0\|_{1,\infty,\Omega} &\le \overline{E}^{1/2} + h^{1-d/p}, \\ \|u_h - U_0\|_{1,\infty,\Omega} &\le C\left(\overline{E}^{1/2} + h^{1-d/p}\right). \end{aligned} \tag{3.11}$$

Furthermore, if, for all $v_1, v_2 \in \mathcal{U}_h \cap W^{1,\infty}(\Omega)$ and $w \in \mathcal{U}_h$ with $\|v_1\|_{1,\infty,\Omega} + \|v_2\|_{1,\infty,\Omega} \le M$, there is a constant $\zeta_0(M)$, with $0 < \zeta_0 < 1$, such that

$$|E(v_1, w) - E(v_2, w)| \le \zeta_0(M) \|v_1 - v_2\|_{1,\Omega} \|w\|_{1,\Omega}, \tag{3.12}$$

then this solution u_h is locally unique.

Proof. We define the nonlinear mapping $\mathcal{L} : \mathcal{U}_h \to \mathcal{U}_h$ by

$$\hat{A}(U_0; \mathcal{L}(v), w) = \hat{A}(U_0; U_0, w) - \mathcal{R}(U_0, v, w) + A(v, w) - A_h(v, w) \qquad \forall w \in \mathcal{U}_h.$$

This mapping is continuous using (3.1) and (3.2). We also define the set

$$B = \left\{ v \in \mathcal{U}_h : \|v - P_h U_0\|_{1,\infty,\Omega} \le \overline{E}^{1/2} + h^{1-d/p} \right\}.$$

Note that, by (3.5),

$$\hat{A}(U_0; \mathcal{L}(v) - P_h U_0, w) = -\mathcal{R}(U_0, v, w) + A(v, w) - A_h(v, w) \qquad \forall w \in \mathcal{U}_h.$$

Choosing $w = G_h^x$ in this equation and applying (3.2), (3.7), (3.9), and (3.10), we see that, with $v \in B$ and $\|v\|_{1,\infty,\Omega} \le M$,

$$\begin{aligned} \|\mathcal{L}(v) - P_h U_0\|_{1,\infty,\Omega} &\le C(M) \left(\|U_0 - v\|_{1,\infty,\Omega}^2 + \overline{E} \right) |\ln h| \\ &\le C(M) \left(\|U_0 - P_h U_0\|_{1,\infty,\Omega}^2 + \|P_h U_0 - v\|_{1,\infty,\Omega}^2 + \overline{E} \right) |\ln h| \\ &\le C(M) \left(h^{2-2d/p} + \overline{E} \right) |\ln h|. \end{aligned}$$

Because $v \in B$ and $\overline{E}$ is bounded (e.g., $\overline{E} \le \mathcal{C}_1$), we see that

$$\begin{aligned} \|v\|_{1,\infty,\Omega} &\le \|v - P_h U_0\|_{1,\infty,\Omega} + \|P_h U_0\|_{1,\infty,\Omega} \\ &\le \overline{E}^{1/2} + C(U_0) \le \mathcal{C}_1^{1/2} + C(U_0) \equiv \mathcal{C}_0. \end{aligned} \tag{3.13}$$

Combining these two inequalities, we have

$$\|\mathcal{L}(v) - P_h U_0\|_{1,\infty,\Omega} \le C(\mathcal{C}_0) \left(h^{2-2d/p} + \overline{E} \right) |\ln h|.$$

Defining $C_1 = 1/C(\mathcal{C}_0)$, it follows from (3.10) that

$$\|\mathcal{L}(v) - P_h U_0\|_{1,\infty,\Omega} \le \overline{E}^{1/2} + C(\mathcal{C}_0) h^{2-2d/p} |\ln h|.$$

Thus there is a constant h_2 such that for $0 < h \le h_2$, we obtain

$$\|\mathcal{L}(v) - P_h U_0\|_{1,\infty,\Omega} \le \overline{E}^{1/2} + h^{1-d/p}.$$

Set $h_1 = \min(h_0, h_2)$. Then, for $0 < h \le h_1$, we see that $\mathcal{L}(B) \subset B$. The Brouwer fixed point theorem means that there is a $u_h \in B$ such that $\mathcal{L}(u_h) = u_h$.

To prove the uniqueness, let u_h^1 and u_h^2 be two solutions of (2.8). Then it follows from (3.3) that

$$\begin{aligned}\hat{A}(U_0; u_h^2 - u_h^1, w) &= \hat{A}(U_0; U_0 - u_h^1, w) - \hat{A}(U_0; U_0 - u_h^2, w) \\ &= \mathcal{R}(U_0, u_h^1, w) - \mathcal{R}(U_0, u_h^2, w) + E(u_h^1, w) - E(u_h^2, w).\end{aligned} \tag{3.14}$$

Because both u_h^1 and u_h^2 are in the set B, it follows from (3.13) that $\|u_h^1\|_{1,\infty,\Omega} + \|u_h^2\|_{1,\infty,\Omega} \le 2\mathcal{C}_0$, which, together with (3.1), (3.12), (3.14), and Poincaré's inequality, implies that

$$\|u_h^1 - u_h^2\|_{1,\Omega} \le \zeta_0(2\mathcal{C}_0)\|u_h^1 - u_h^2\|_{1,\Omega}.$$

Since $\zeta_0 < 1$ via assumption, then $u_h^1 = u_h^2$. Therefore, the solution u_h is locally unique. ◻

4. Improved error estimates for nonlinear problems. The main result in this section is stated in the next theorem. The multiscale finite element solution u_h used below refers to the one that satisfies the conditions in Theorem 3.1. To avoid unnecessary techniques, we perform an error analysis when system (2.7) is replaced by

$$\begin{aligned} &(a(x_T, x/\epsilon, v(x_T))\nabla R_T(v), \nabla w)_T = 0 && \forall w \in H_0^1(T), \\ &R_T(v) = v && \text{on } \partial T, \end{aligned} \tag{4.1}$$

where x_T is any point in $T \in T_h$ (e.g., the barycenter of T).

THEOREM 4.1. *Let U_0 and u_h be the solutions of* (2.5) *and* (2.8), *respectively, and $U_0 \in W^{2,\infty}(\Omega)$. Then there is $h_0 > 0$ such that for $0 < h < h_0$,*

$$\|U_0 - u_h\|_{1,\Omega} \le C\left(h + \sqrt{\frac{\epsilon}{h}}\right), \quad \|U_0 - u_h\|_{1,\infty,\Omega} \le C\left(h + \sqrt{\frac{\epsilon}{h}}\right)|\ln h|, \tag{4.2}$$

provided that ϵ/h is sufficiently small.

This theorem can be shown by combining the next two propositions.

PROPOSITION 4.2. *In addition to the assumptions of Theorem* 3.1, *let $U_0 \in W^{2,\infty}(\Omega)$. Then there are constants C_2 (see the proof below) and $h_0 > 0$ such that if $C_1 < C_2$ and $0 < h < h_0$, then*

$$\|U_0 - u_h\|_{1,\Omega} \le C\left(h + \overline{E}\right), \quad \|U_0 - u_h\|_{1,\infty,\Omega} \le C\left(h + \overline{E}\right)|\ln h|. \tag{4.3}$$

PROPOSITION 4.3. *Under the assumptions of Theorem* 4.1, *if $\sqrt{\epsilon/h}|\ln h|$ is sufficiently small, then conditions* (3.10) *and* (3.12) *hold. Moreover,*

$$\overline{E} \le C\sqrt{\frac{\epsilon}{h}}. \tag{4.4}$$

Proof of Proposition 4.2. Taking $w = P_h U_0 - u_h$ in (3.3) and using (3.1), (3.2), and (3.6), we see that

$$\|P_h U_0 - u_h\|_{1,\Omega} \le C\left(\|U_0 - u_h\|_{1,4,\Omega}^2 + \overline{E} + h\right). \tag{4.5}$$

Applying an interpolation inequality, we have

$$\|U_0 - u_h\|_{1,4,\Omega}^2 \le \|U_0 - u_h\|_{1,\Omega} \|U_0 - u_h\|_{1,\infty,\Omega}, \tag{4.6}$$

which, together with (3.6) and (3.13), yields

$$\|P_h U_0 - u_h\|_{1,\Omega} \le C_3 \left(\overline{E}^{1/2} + h^{1-d/p}\right) \|P_h U_0 - u_h\|_{1,\Omega} + C\left(h + \overline{E}\right).$$

Choosing $w = G_h^x$ in (3.3) and using (3.2), (3.6), and (3.9), we have

$$\begin{aligned}\|P_h U_0 - u_h\|_{1,\infty,\Omega} &\le C_4(\overline{E} + h)|\ln h| \|P_h U_0 - u_h\|_{1,\infty,\Omega} \\ &\quad + C\left(\|U_0 - u_h\|_{1,\infty,\Omega}^2 + \overline{E} + h\right)|\ln h|.\end{aligned}$$

From (3.6), (3.7), and (3.10) it follows that

$$\begin{aligned}\|P_h U_0 - u_h\|_{1,\infty,\Omega} &\le C_4(\overline{E}^{1/2} + h^{1-d/p})|\ln h| \|P_h U_0 - u_h\|_{1,\infty,\Omega} \\ &\quad + C\left(\overline{E} + h\right)|\ln h|.\end{aligned}$$

Now, we set

$$C_2 = \min\left(\frac{|\ln h|}{2C_3}, \frac{1}{2C_4}\right)$$

and choose h_0 such that $\overline{E}^{1/2}|\ln h| \le C_2$. With these two choices, we obtain

$$\begin{aligned}\|P_h U_0 - u_h\|_{1,\Omega} &\le C\left(\overline{E} + h\right), \\ \|P_h U_0 - u_h\|_{1,\infty,\Omega} &\le C\left(\overline{E} + h\right)|\ln h|,\end{aligned}$$

which, together with (3.6), gives the desired result. ∎

Proposition 4.3 can be proven from the next three lemmas.

LEMMA 4.4. *For $v, w \in \mathcal{U}_h$, we have*

$$|R(v) - R(w)|_{1,T} \le C\left(\|v - w\|_{0,\infty,T}\left[|v|_{1,T} + |w|_{1,T}\right] + |v - w|_{1,T}\right), \quad T \in T_h. \tag{4.7}$$

Proof. It follows from (2.2) and (4.1) that

$$\begin{aligned}C|R(v) - R(w)|_{1,T}^2 &\le \left(a\left(x_T, \frac{x}{\epsilon}, v(x_T)\right)\nabla[R(v) - R(w)], \nabla[R(v) - R(w)]\right)_T \\ &\equiv J_1 + J_2,\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}J_1 &= \left(a\left(x_T, \frac{x}{\epsilon}, v(x_T)\right)\nabla R(v) - a\left(x_T, \frac{x}{\epsilon}, w(x_T)\right)\nabla R(w), \nabla(v - w)\right)_T, \\ J_2 &= -\left(\left[a\left(x_T, \frac{x}{\epsilon}, v(x_T)\right) - a\left(x_T, \frac{x}{\epsilon}, w(x_T)\right)\right]\nabla R(w), \nabla[R(v) - R(w)]\right)_T.\end{aligned}$$

Note that

$$\begin{aligned}J_1 &= \left(\left[a\left(x_T, \frac{x}{\epsilon}, v(x_T)\right) - a\left(x_T, \frac{x}{\epsilon}, w(x_T)\right)\right]\nabla R(v), \nabla(v - w)\right)_T \\ &\quad + \left(a\left(x_T, \frac{x}{\epsilon}, w(x_T)\right)\nabla[R(v) - R(w)], \nabla(v - w)\right)_T.\end{aligned}$$

As a result, using (2.2), we bound J_1 and J_2 as follows:

$$|J_1| \le C \left(\|v-w\|_{0,\infty,T}|R(v)|_{1,T} + |R(v)-R(w)|_{1,T}\right)|v-w|_{1,T},$$
$$|J_2| \le C\|v-w\|_{0,\infty,T}|R(w)|_{1,T}|R(v)-R(w)|_{1,T}.$$

Substituting these two bounds for J_1 and J_2 into (4.8) and using (3.4) implies the desired result (4.7). ∎

For $v, w \in \mathcal{U}_h$, we define

$$Q(v) = v(x) + \epsilon \sum_{k=1}^{d} \chi^k \left(x_T, \frac{x}{\epsilon}, v(x_T)\right) \frac{\partial v}{\partial x_k}(x),$$
$$Q(w) = w(x) + \epsilon \sum_{k=1}^{d} \chi^k \left(x_T, \frac{x}{\epsilon}, w(x_T)\right) \frac{\partial w}{\partial x_k}(x).$$

Set $\theta_\epsilon(v) = R(v) - Q(v)$ and $\theta_\epsilon(w) = R(w) - Q(w)$. Note that

$$\begin{aligned} &-\nabla \cdot \left(a\left(x_T, \frac{x}{\epsilon}, v(x_T)\right)\nabla\theta_\epsilon(v)\right) = 0 &&\text{in } T,\\ &\theta_\epsilon(v) = -\epsilon\sum_{k=1}^{d} \chi^k\left(x_T, \frac{x}{\epsilon}, v(x_T)\right)\frac{\partial v}{\partial x_k} &&\text{on } \partial T, \end{aligned} \tag{4.9}$$

and an analogous definition can be given for $\theta_\epsilon(w)$. These two quantities satisfy [5, 17]

$$|\theta_\epsilon(v)|_{1,T} \le C\sqrt{\frac{\epsilon}{h}}\;|v|_{1,T}, \quad |\theta_\epsilon(w)|_{1,T} \le C\sqrt{\frac{\epsilon}{h}}\;|w|_{1,T}, \tag{4.10}$$

which hold because ∇v and ∇w are piecewise constant.

LEMMA 4.5. *For $v, w \in \mathcal{U}_h$, we have*

(4.11)
$$|\theta_\epsilon(v) - \theta_\epsilon(w)|_{1,T} \le C\sqrt{\frac{\epsilon}{h}}\left(\|v-w\|_{0,\infty,T}\left[|v|_{1,T} + |w|_{1,T}\right] + |v-w|_{1,T}\right), \quad T \in T_h.$$

Proof. Let $\xi_\epsilon \in C_0^\infty(T)$, $0 \le \xi_\epsilon \le 1$, be a cut-off function in T such that $\xi_\epsilon = 1$ outside a ϵ-neighborhood of the boundary ∂T and $|\nabla \xi_\epsilon| \le C\epsilon^{-1}$ with C independent of ϵ and T. Define, for $v \in \mathcal{U}_h$,

$$\varphi_\epsilon(v) = \theta_\epsilon(v) + \epsilon\sum_{k=1}^{d} \chi^k\left(x_T, \frac{x}{\epsilon}, v(x_T)\right)\frac{\partial v}{\partial x_k}(1-\xi_\epsilon),$$

and a similar meaning can be given for $\varphi_\epsilon(w)$, $w \in \mathcal{U}_h$. Calculations show that

$$\begin{aligned} &C|\varphi_\epsilon(v) - \varphi_\epsilon(w)|_{1,T}\\ &\quad\le \max_{x\in T}\left|a\big(x_T, x/\epsilon, v(x_T)\big) - a\big(x_T, x/\epsilon, w(x_T)\big)\right| \left(|\varphi_\epsilon(v) - \theta_\epsilon(v)|_{1,T} + |\varphi_\epsilon(w)|_{1,T}\right)\\ &\qquad + \max_{x\in T}\left|a\big(x_T, x/\epsilon, w(x_T)\big)\right|\, |\varphi_\epsilon(v) - \theta_\epsilon(v) - \varphi_\epsilon(w) + \theta_\epsilon(w)|_{1,T} \end{aligned}$$

and

$$|\varphi_\epsilon(v) - \theta_\epsilon(v)|_{1,T} \le C\sqrt{\frac{\epsilon}{h}}|v|_{1,T}, \quad |\varphi_\epsilon(w) - \theta_\epsilon(w)|_{1,T} \le C\sqrt{\frac{\epsilon}{h}}|w|_{1,T}.$$

Also, using (4.10), we see that

$$|\varphi_\epsilon(w)|_{1,T} \le |\varphi_\epsilon(w) - \theta_\epsilon(w)|_{1,T} + |\theta_\epsilon(w)|_{1,T} \le C\sqrt{\frac{\epsilon}{h}}|w|_{1,T}.$$

Observe that

$$\begin{aligned}&\varphi_\epsilon(v) - \theta_\epsilon(v) - \varphi_\epsilon(w) + \theta_\epsilon(w)\\ &\quad = \epsilon\sum_{k=1}^d \left(\chi^k\left(x_T, \frac{x}{\epsilon}, v(x_T)\right) - \chi^k\left(x_T, \frac{x}{\epsilon}, w(x_T)\right)\right)\frac{\partial v}{\partial x_k}(1-\xi_\epsilon)\\ &\qquad + \epsilon\sum_{k=1}^d \chi^k\left(x_T, \frac{x}{\epsilon}, w(x_T)\right)\frac{\partial(v-w)}{\partial x_k}(1-\xi_\epsilon).\end{aligned}$$

Consequently, by the continuity of $\{\chi^k\}_{k=1}^d$, we see that

$$|\varphi_\epsilon(v) - \theta_\epsilon(v) - \varphi_\epsilon(w) + \theta_\epsilon(w)|_{1,T} \le C\sqrt{\frac{\epsilon}{h}}\left(\|v-w\|_{0,\infty,T}|v|_{1,T} + |v-w|_{1,T}\right).$$

Combining these inequalities gives the desired result (4.11). ∎

The next lemma indicates that $E(\cdot,\cdot)$ has certain continuity with respect to its first argument.

LEMMA 4.6. *For $v_1, v_2, w \in \mathcal{U}_h$ satisfying $\|v_1\|_{1,\infty,\Omega} + \|v_2\|_{1,\infty,\Omega} \le M$, we have*

$$|E(v_1,w) - E(v_2,w)| \le C(M)\sqrt{\frac{\epsilon}{h}}\|v_1 - v_2\|_{1,\Omega}|w|_{1,\Omega}, \quad T \in T_h. \tag{4.12}$$

Proof. Define $l = [h/\epsilon]$, and let $I_{l\epsilon}$ be the cube of size $l\epsilon$ at x_T. By the definition of $R(v_1)$ and the relation that $R(v_1) = \theta_\epsilon(v_1) + Q(v_1)$, we see that

$$\begin{aligned}\big(a(x_T, x/\epsilon, v_1(x_T))\nabla R(v_1), \nabla R(w)\big)_T &= \big(a(x_T, x/\epsilon, v_1(x_T))\nabla R(v_1), \nabla w\big)_T\\ &= \big(a(x_T, x/\epsilon, v_1(x_T))\nabla[\theta_\epsilon(v_1) + Q(v_1)], \nabla w\big)_T.\end{aligned}$$

Also, note that

$$\frac{1}{|I_{l\epsilon}|}\left(a\left(x_T, \frac{x}{\epsilon}, v_1(x_T)\right)\nabla Q(v_1), \nabla w\right)_{I_{l\epsilon}} = \nabla w\cdot\mathcal{A}\big(x_T, v_1(x_T)\big)\nabla v_1.$$

Then $E(v_1,w) - E(v_2,w)$ can be split as follows:

$$E(v_1,w) - E(v_2,w) = \sum_{T\in T_h}\left(E(v_1,w) - E(v_2,w)\right)|_T = \sum_{T\in T_h}(J_3 + J_4), \tag{4.13}$$

where

$$\begin{aligned}J_3 &= \left(a\left(x_T, \frac{x}{\epsilon}, v_1(x_T)\right)\nabla\theta_\epsilon(v_1) - a\left(x_T, \frac{x}{\epsilon}, v_2(x_T)\right)\nabla\theta_\epsilon(v_2), \nabla w\right)_T,\\ J_4 &= \left(a\left(x_T, \frac{x}{\epsilon}, v_1(x_T)\right)\nabla Q(v_1), \nabla w\right)_T\\ &\quad - \frac{|T|}{|I_{l\epsilon}|}\left(a\left(x_T, \frac{x}{\epsilon}, v_1(x_T)\right)\nabla Q(v_1), \nabla w\right)_{I_{l\epsilon}}\\ &\quad - \left(a\left(x_T, \frac{x}{\epsilon}, v_2(x_T)\right)\nabla Q(v_2), \nabla w\right)_T\\ &\quad - \frac{|T|}{|I_{l\epsilon}|}\left(a\left(x_T, \frac{x}{\epsilon}, v_2(x_T)\right)\nabla Q(v_2), \nabla w\right)_{I_{l\epsilon}}.\end{aligned}$$

It follows from Lemma 4.5 that

$$|J_3| \le C\sqrt{\frac{\epsilon}{h}}\big(\|v_1 - v_2\|_{0,\infty,T}\,[|v_1|_{1,T} + |v_2|_{1,T}] + |v_1 - v_2|_{1,T}\big)|w|_{1,T}. \tag{4.14}$$

The term J_4 can be rewritten as follows:

$$\begin{aligned} J_4 = &-\left(\frac{|T|}{|l\epsilon|^d} - 1\right)\left(a\left(x_T, \frac{x}{\epsilon}, v_1(x_T)\right)\nabla Q(v_1) - a\left(x_T, \frac{x}{\epsilon}, v_2(x_T)\right)\nabla Q(v_2), \nabla w\right)_T \\ &+ \frac{|T|}{|l\epsilon|^d}\left(a\left(x_T, \frac{x}{\epsilon}, v_1(x_T)\right)\nabla Q(v_1) - a\left(x_T, \frac{x}{\epsilon}, v_2(x_T)\right)\nabla Q(v_2), \nabla w\right)_{T\setminus I_{l\epsilon}}. \end{aligned}$$

Then it can be bounded by

$$|J_4| \le C\frac{\epsilon}{h}\left(\|v_1 - v_2\|_{0,\infty,T}\,[|v_1|_{1,T} + |v_2|_{1,T}] + |v_1 - v_2|_{1,T}\right)|w|_{1,T}. \tag{4.15}$$

Applying an inverse inequality, we see that

$$\begin{aligned} \|v_1 - v_2\|_{0,\infty,T}|v_1|_{1,T} &\le Ch_T^{-d/2}\|v_1 - v_2\|_{0,T}h_T^{d/2}\|v_1\|_{1,\infty,T} \\ &= C\|v_1 - v_2\|_{0,T}\|v_1\|_{1,\infty,T}, \end{aligned} \tag{4.16}$$

since $\nabla v_1|_T$ is a constant. A similar result holds for $\|v_1 - v_2\|_{0,\infty,T}|v_2|_{1,T}$. Substituting (4.14)–(4.16) into (4.13) implies the desired result. ∎

Proof of Proposition 4.3. Taking $v_2 = 0$ in (4.12) generates (4.4). Also, if $\sqrt{\epsilon/h}|\ln h|$ is sufficiently small, $\overline{E}^{1/2}|\ln h|$ can be made smaller than any given threshold, and so (3.10) holds. Finally, choose $\zeta_0(M) = C(M)\sqrt{\epsilon/h}$. Then if $\sqrt{\epsilon/h}|\ln h|$ is sufficiently small, $\zeta_0(M) < 1$, and thus (3.12) is verified. ∎

5. An oversampling technique. Note that estimates (4.2) deteriorate when ϵ is of the same order as the mesh size h. This phenomenon reveals a "resonance error" between the grid scale h and the scale ϵ of the continuous problem (2.1). This resonance is due to a mismatch between the local solution of (2.7) and the global solution of (2.1) on the boundary of each $T \in T_h$, which produces a boundary layer. Since this layer is thin, we can sample in a (local) domain with size larger than h and utilize only the interior sampled information. In this manner, the influence of the boundary layer in the larger domain can be greatly reduced. In this section, we extend an oversampling technique for linear problems [13, 16] to the nonlinear problem (2.1) in order to reduce the resonance error in (4.2).

For each $T \in T_h$, we indicate by $S(T)$ a macroelement which contains T and satisfies the following condition: There are positive constants C_3 and C_4, independent of h and ϵ, such that $h_S \le C_3 h_T$ and $\text{dist}(\partial S, \partial T) \ge C_4 h_T$. For each $v \in \mathcal{U}_h(T)$, we extend it to $\mathcal{U}_h(S)$ as follows. Let $\{\phi_i^T\}_1^{d+1}$ and $\{\psi_i^S\}_1^{d+1}$ be the respective bases of $\mathcal{U}_h(T)$ and $\mathcal{U}_h(S)$. Set

$$v|_T = \sum_{i=1}^{d+1} c_i^T\phi_i^T, \quad \phi_i^T = \sum_{j=1}^{d+1} c_{ij}^T\psi_j^S|_T.$$

Then we define $\widehat{v} \in \mathcal{U}_h(S)$ by

$$\widehat{v} = \sum_{i,j=1}^{d+1} c_i^T c_{ij}^T \psi_j^S.$$

Now, for any $v \in \mathcal{U}_h$, we define $R_S(v) \in H^1(S)$, $T \subset S$, $T \in T_h$, by

$$(5.1) \qquad \begin{aligned} &(a(x, x/\epsilon, \widehat{v})\nabla R_S(v), \nabla w)_S = 0 && \forall w \in H_0^1(S), \\ &R_S(v) = \widehat{v} && \text{on } \partial S. \end{aligned}$$

The global operator R is defined by

$$R(v)|_T = R_S(v)|_T \qquad \forall v \in \mathcal{U}_h,\ T \in T_h.$$

The oversampled MsFEM for (2.1) is to seek $u_h \in \mathcal{U}_h$ such that

$$(5.2) \qquad \sum_{T \in T_h} \left(a(x, x/\epsilon, u_h)\nabla R_{S(T)}(u_h), \nabla R_{S(T)}(v)\right)_T = (f, v) \qquad \forall v \in \mathcal{U}_h.$$

The existence and uniqueness of a solution to (5.2) can be shown in Theorem 3.1. Furthermore, combining the error analysis in the previous section and that for the oversampled MsFEM for linear problems given in [5, 13], the following improved error estimates can be shown as in Theorem 4.1.

Theorem 5.1. *Let U_0 and u_h be the solutions of* (2.5) *and* (5.2)*, respectively, and $U_0 \in W^{2,\infty}(\Omega)$. Then there is $h_0 > 0$ such that for $0 < h < h_0$,*

$$(5.3) \qquad \|U_0 - u_h\|_{1,\Omega} \le C\left(h + \frac{\epsilon}{h}\right), \quad \|U_0 - u_h\|_{1,\infty,\Omega} \le C\left(h + \frac{\epsilon}{h}\right)|\ln h|,$$

provided that ϵ/h is sufficiently small.

Note that while these estimates improve those in (5.15), resonance persists.

6. Approximation to u^ϵ.

Theorems 4.1 and 5.1 show that the multiscale finite element solution u_h of (2.8) is a good approximation of the macroscopic solution U_0. We now consider an approximation to the solution u^ϵ of (2.1).

Define $R(u_h)$ as the solution of (4.1) with v replaced by u_h:

$$(6.1) \qquad \begin{aligned} &(a(x_T, x/\epsilon, u_h(x_T))\nabla R_T(u_h), \nabla w)_T = 0 && \forall w \in H_0^1(T), \\ &R_T(v) = u_h && \text{on } \partial T. \end{aligned}$$

Applying (2.4), $R_T(u_h)$ satisfies

$$(6.2) \qquad \nabla R_T(u_h) = \nabla u_h + \sum_{k=1}^{d} \nabla_y \chi^k \left(x_T, \frac{x}{\epsilon}, u_h(x_T)\right) \frac{\partial u_h}{\partial x_k} + \nabla\theta_\epsilon(u_h), \qquad T \in T_h.$$

Also, define the first order approximation of u_1^ϵ by

$$u_1^\epsilon = U_0 + \epsilon \sum_{k=1}^{d} \chi^k \left(x, \frac{x}{\epsilon}, U_0(x)\right) \frac{\partial U_0}{\partial x_k}.$$

Clearly,

$$(6.3) \qquad \nabla u_1^\epsilon = \nabla U_0 + \sum_{k=1}^{d} \left\{ \left(\epsilon\nabla\chi^k + \nabla_y\chi^k\right) \frac{\partial U_0}{\partial x_k} + \epsilon \frac{\partial \chi^k}{\partial U_0} \frac{\partial U_0}{\partial x_k} \nabla U_0 + \epsilon\chi^k \nabla \frac{\partial U_0}{\partial x_k} \right\}.$$

Combining (6.2) and (6.3) gives

(6.4)
$$
\begin{aligned}
|u_1^\epsilon - R(u_h)|_{1,T} &\le C\left(|U_0 - u_h|_{1,T} + \epsilon\left[\|U_0\|_{2,T} + \|U_0\|_{1,4,T}^2\right]\right) + |\theta_\epsilon(u_h)|_{1,T} \\
&\quad + \left\|\sum_{k=1}^d \left[\nabla_y \chi^k\left(x_T, \frac{x}{\epsilon}, u_h(x_T)\right) - \nabla_y \chi^k\left(x, \frac{x}{\epsilon}, U_0(x)\right)\right] \frac{\partial U_0}{\partial x_k}\right\|_{0,T} \\
&\le C\Big\{|U_0 - u_h|_{1,T} + |U_0 - u_h|_{1,\infty,T}|U_0|_{1,T} + |\theta_\epsilon(u_h)|_{1,T} \\
&\quad + (\epsilon + h)\left(\|U_0\|_{2,T} + \|U_0\|_{1,4,T}^2 + |U_0|_{1,\infty,T}|U_0|_{1,T}\right)\Big\}.
\end{aligned}
$$

A classical estimate for $u^\epsilon - u_1^\epsilon$ [2, 3, 22] gives

$$\|u^\epsilon - u_1^\epsilon\|_{1,\Omega} \le C\sqrt{\epsilon}. \tag{6.5}$$

Finally, combining (4.2), (4.10), (6.4), and (6.5), we obtain the next theorem.

THEOREM 6.1. *Let u^ϵ be the solution of* (2.1)*, $R(u_h)$ be defined by* (6.1)*, and $U_0 \in W^{2,\infty}(\Omega)$. Then there is $h_0 > 0$ such that for $0 < h < h_0$,*

$$|u^\epsilon - R(u_h)|_{1,\Omega} \le C\left(h + \sqrt{\epsilon} + \sqrt{\frac{\epsilon}{h}}\right)|\ln h|, \tag{6.6}$$

provided that ϵ/h is sufficiently small.

Note that estimate (6.6) is similar to (2.9) obtained for a linear counterpart of (2.1). The oversampling technique discussed in the previous section can also be applied to (6.1).

7. A random homogenization problem. In the previous sections we have assumed that the coefficient a_ϵ in (2.1) is periodic. In many problems such as in porous media flows [6], this coefficient is often random. In this section we extend the multiscale finite element analysis performed for the nonlinear problem (2.1) to a multiscale problem with a random coefficient.

Let (D, F, P) be a probability space and $a(y,\omega) = \big(a_{ij}(y,\omega)\big)$ be a random field, $y \in \mathbb{R}^d$, $\omega \in D$, whose statistics is invariant under integer shifts. Furthermore, let a satisfy the uniform ellipticity condition (2.2); i.e.,

$$a_*|\zeta|^2 \le \sum_{i,j=1}^d a_{ij}(y,\omega)\zeta_i\zeta_j \le a^*|\zeta|^2 \qquad \forall \omega \in D,\ y,\ \zeta \in \mathbb{R}^d, \tag{7.1}$$

for some positive constants a_* and a^*. Problem (2.1) now takes the form

$$
\begin{aligned}
-\nabla \cdot \big(a(x/\epsilon,\omega)\nabla u^\epsilon\big) &= f \qquad \text{in } \Omega, \\
u^\epsilon &= 0 \qquad \text{on } \Gamma.
\end{aligned} \tag{7.2}
$$

7.1. Homogenization results. We collect some homogenization results for problem (7.2), following [21]. As in (2.4), let χ^j satisfy [18]

$$-\nabla_y \cdot \big(a(y,\omega)\nabla_y \chi^j\big) = \sum_{i=1}^d \frac{\partial}{\partial y_i} a_{ij}(y,\omega), \tag{7.3}$$

and $\nabla\chi^j$ is assumed to be stationary under integer shifts. χ^j is generally not stationary.

Define the average operator with respect to the measure P (mathematical expectation)

$$\langle v\rangle = \mathbb{E}\int_{[0,1]^d} v(y)\ dy.$$

Also, define

$$[v]_m \equiv [v;m] = \frac{1}{m^d}\int_{[0,m]^d} v(y)\ dy.$$

The homogenized coefficient $\mathcal{A}$ is given by

$$\mathcal{A} = \langle a(\mathcal{I}+\nabla\chi)\rangle\,, \tag{7.4}$$

where $\mathcal{I}$ is the identity matrix and $\chi = (\chi^1, \chi^2, \ldots, \chi^d)^T$.

For any $\rho > 0$, we consider the auxiliary problem

$$-\nabla_y\cdot(a(y,\omega)\nabla_y u) + \rho u = \sum_{i=1}^{d}\frac{\partial g_i}{\partial y_i}, \tag{7.5}$$

where

$$g_i \in \{v : \langle v^2\rangle \le G^2\}, \qquad i = 1, 2, \ldots, d,$$

with v a random field whose statistics is stationary under integer shifts.

For each fixed realization of $\{a(y,\cdot)\}$, let η_x be the diffusion process generated by $-\nabla_y\cdot(a(y,\omega)\nabla_y)$ and starting from x at $t = 0$, and let M_x be the expectation with respect to η_x. Set

$$\Gamma(s) = \int_0^s e^{-\rho\tau}\sum_{i=1}^{d}\frac{\partial g_i}{\partial y_i}\big(\eta(\tau)\big)\ d\tau.$$

It is known [14] that the solution of problem (7.5) is

$$u_\rho(x) = M_x\Gamma(\infty).$$

Lemmas 7.1–7.3 and 7.5 below can be found in [21] and Lemma 7.4 in [10]. Note that the homogenization results in [21] may be overestimated because they are based on the Green function estimates that are not required for the computation of effective coefficients. Because of this, the convergence result here may be overestimated as well.

LEMMA 7.1. *For the solution u_ρ of* (7.5)*, there are constants $C > 0$, independent of ρ, such that*

$$\begin{aligned}
&\left\langle|\nabla u_\rho|^2\right\rangle + \rho\left\langle u_\rho^2\right\rangle \le C\left\langle|g|^2\right\rangle,\\
&\left\langle\left(M_x\Gamma(\infty)\right)^2\right\rangle^{1/2} \le \frac{CG^2}{\rho},\\
&\left\langle M_x\left(\Gamma(\infty)-\Gamma(s)\right)^2\right\rangle \le \frac{CG^2}{\rho}e^{-2s\rho},
\end{aligned} \tag{7.6}$$

where $g = (g_1, g_2, \ldots, g_d)^T$.

Due to the presence of the lower order term ρu, the Green function associated with the differential operator $-\nabla_y \cdot (a(y,\omega)\nabla_y) + \rho$ in the left-hand side of (7.5) decays exponentially with a rate of order $\mathcal{O}(\sqrt{\rho})$. To be specific, define

$$B_\rho = \left\{ x \in \mathbb{R}^d : \|x\|_\infty \le \rho^{-\frac{1}{2}\left|\ln \rho^{-1}\right|^{1/2}} \right\},$$

where $\|x\|_\infty = \max\{|x_i|, i = 1, 2, \ldots, d\}$.

LEMMA 7.2. *Let t be the first exit time of B_ρ starting at $x \in B_\rho$, and define $\hat{u}_\rho(x) = M_x \Gamma(t)$. Then, if ρ is small enough,*

$$\mathbb{E} \int_{\|x\|_\infty \le 10} |u_\rho(x) - \hat{u}_\rho(x)|^2 \, dx \le CG^2 e^{-C|\ln \rho^{-1}|^2},$$
$$\mathbb{E} \int_{\|x\|_\infty \le 1} |\nabla u_\rho(x) - \nabla \hat{u}_\rho(x)|^2 \, dx \le CG^2 e^{-C|\ln \rho^{-1}|^2}.$$

LEMMA 7.3. *Let $\{a^1, g^1\}$ and $\{a^2, g^2\}$ be two sets of data satisfying*

$$\{a^1(y), g^1(y)\} = \{a^2(y), g^2(y)\}, \qquad y \notin B,$$

where $B \subset \mathbb{R}^d$ is a domain, and u_ρ^1 and u_ρ^2 be the respective solutions of (7.5) *associated with $\{a^1, g^1\}$ and $\{a^2, g^2\}$. Then*

$$\int_{\mathbb{R}^d} |u_\rho^1(x) - u_\rho^2(x)|^2 \, dx \le \frac{C}{\rho} \int_{\mathbb{R}^d} \left(G^2 + |\nabla u_\rho^1(x)|^2\right) I_B(x) \, dx,$$

where I_B is the indicator function of B.

For a subdomain $B \subset \mathbb{R}^d$, denote by $\Phi(B)$ the σ-algebra generated by the parameters $\{a(y,\omega) : y \in B\}$. Let ζ_1 and ζ_2 be two random variables that are measurable with respect to $\Phi(B_1)$ and $\Phi(B_2)$, respectively. We will use the exponential decay condition

$$|\mathbb{E}(\zeta_1 \zeta_2) - \mathbb{E}(\zeta_1)\mathbb{E}(\zeta_2)| \le e^{-C \operatorname{dist}(B_1, B_2)} \sqrt{\mathbb{E}\zeta_1^2} \sqrt{\mathbb{E}\zeta_2^2}. \tag{7.7}$$

This type of exponential decay condition is often used for geostatistical models [15].

LEMMA 7.4. *Under condition* (7.7), *we have*

$$\mathbb{E}\left[u_\rho; m\right]^2 \le C \left(\frac{G^2}{\rho} \left(\frac{|\ln \rho^{-1}|^2}{\rho^{1/2} m} \right) + e^{-C|\ln \rho^{-1}|^2} \right).$$

LEMMA 7.5. *Let $\chi_\rho = \left(\chi_\rho^1, \chi_\rho^2, \ldots, \chi_\rho^d\right)^T$, where χ_ρ^i is the solution of* (7.5) *with*

$$g = \left(a_{i1}, a_{i2}, \ldots, a_{id}\right)^T.$$

Under condition (7.7), *for any $0 < \lambda < 1/2$, we have*

$$|\mathcal{A} - \langle a(\mathcal{I} + \nabla \chi_\rho) \rangle| \le C\rho^{(d-2-2\lambda)/(4+d)},$$

where $|\cdot|$ represents a matrix norm.

7.2. The MsFEM. Note that the definition of the MsFEM (2.8) does not utilize any periodicity or macroscopic model. Thus, in the random case it can be defined in the same manner as in the periodic case. That is, for any $v \in \mathcal{U}_h$, we define its local solution $R_T(v) \in H^1(T)$, $T \in T_h$, by

$$\begin{aligned} &(a_\epsilon \nabla R_T(v), \nabla w)_T = 0 \qquad && \forall w \in H_0^1(T), \\ &R_T(v) = v && \text{on } \partial T. \end{aligned} \tag{7.8}$$

Define

$$A_h(v,w) = \sum_{T \in T_h} \left(a_\epsilon \nabla R_T(v), \nabla R_T(w)\right)_T, \qquad v,\ w \in \mathcal{U}_h.$$

Then the MsFEM for (7.2) is as follows: Find $u_h \in \mathcal{U}_h$ such that

$$A_h(u_h, v) = (f, v) \qquad \forall v \in \mathcal{U}_h. \tag{7.9}$$

THEOREM 7.6. *Let U_0 and u_h be the respective solutions of* (2.5) *and* (7.9)*, where the homogenized coefficient $\mathcal{A}$ is now given by* (7.4)*, and $U_0 \in W^{2,\infty}(\Omega)$. Then, under condition* (7.7)*, we have*

$$\begin{aligned} &\mathbb{E}\ \|U_0 - u_h\|_{1,\Omega} \le C\left(h + \left(\frac{\epsilon}{h}\right)^\kappa\right), \\ &\mathbb{E}\ \|U_0 - u_h\|_{0,\Omega} \le C\left(h^2 + \left(\frac{\epsilon}{h}\right)^\kappa\right), \\ &\mathbb{E}\ \|U_0 - u_h\|_{1,\infty,\Omega} \le C\left(h + \left(\frac{\epsilon}{h}\right)^\kappa\right)|\ln h|, \end{aligned} \tag{7.10}$$

where

$$\kappa = \begin{cases} \dfrac{6 - 12\lambda}{25 - 8\lambda} & \text{if } d = 3, \\ \dfrac{1}{2} & \text{if } d = 1 \end{cases}$$

for any $0 < \lambda < 1/2$.

Note that the case $d = 2$ remains open. This theorem can be proven by combining the next two propositions. As in the nonlinear case in the third section, define

$$\overline{E} = \max_{v,w \in \mathcal{U}_h} \frac{|A_h(v,w) - A(v,w)|}{|v|_{1,\Omega}|w|_{1,\Omega}}.$$

PROPOSITION 7.7. *Let U_0 and u_h be the respective solutions of* (2.5) *and* (7.9)*, and let $U_0 \in W^{2,\infty}(\Omega)$. Then*

$$\begin{aligned} &\|U_0 - u_h\|_{1,\Omega} \le C\left(h + \overline{E}\right), \\ &\|U_0 - u_h\|_{0,\Omega} \le C\left(h^2 + \overline{E}\right). \end{aligned} \tag{7.11}$$

Furthermore, if there is a constant C_5 such that $\overline{E}|\ln h| < C_5$, then there is a constant h_0 such that for any $0 < h < h_0$,

$$\|U_0 - u_h\|_{1,\infty,\Omega} \le C\left(h + \overline{E}\right)|\ln h|. \tag{7.12}$$

PROPOSITION 7.8. *Under condition* (7.7)*, we have*

$$\mathbb{E}\overline{E} \leq C\left(\frac{\epsilon}{h}\right)^{\kappa}, \tag{7.13}$$

where κ *is defined as in Theorem* 7.6.

As in (3.8), for a given $x \in \Omega$, we define the Green function $G^x \in H_0^1(\Omega)$ and its discrete counterpart $G_h^x \in \mathcal{U}_h$ by

$$\begin{aligned} A(G^x, v) &= \partial v(x) \qquad \forall v \in H_0^1(\Omega),\\ A(G_h^x, v) &= \partial v(x) \qquad \forall v \in \mathcal{U}_h. \end{aligned} \tag{7.14}$$

They satisfy

$$\|G^x - G_h^x\|_{1,1,\Omega} \leq C, \quad \|G_h^x\|_{1,1,\Omega} \leq C|\ln h|. \tag{7.15}$$

Below P_h indicates the standard Lagrange interpolation operator from $H_0^1(\Omega)$ into $\mathcal{U}_h$.

Proof of Proposition 7.7. It follows from the first Strang lemma [7] that

$$\|U_0 - u_h\|_{1,\Omega} \leq C \inf_{v \in \mathcal{U}_h}\left(\|U_0 - v\|_{1,\Omega} + \sup_{w \in \mathcal{U}_h} \frac{|A(v,w) - A_h(v,w)|}{\|w\|_{1,\Omega}}\right).$$

Taking $v = P_h U_0$ and using the definition of $\overline{E}$ implies the first inequality in (7.11). The second inequality in (7.11) follows from a standard duality argument [4, 7].

To prove (7.12), using (7.14), we see that

$$\begin{aligned} \partial(U_0 - u_h)(x) &= A(G^x, U_0 - P_h U_0) + A(G^x, P_h U_0 - u_h)\\ &= A(G^x - G_h^x, U_0 - P_h U_0) + A(G_h^x, U_0 - u_h)\\ &= A(G^x - G_h^x, U_0 - P_h U_0) + A_h(u_h, G_h^x) - A(u_h, G_h^x)\\ &= A(G^x - G_h^x, U_0 - P_h U_0) + (A_h(P_h U_0, G_h^x) - A(P_h U_0, G_h^x))\\ &\quad + (A_h(u_h - P_h U_0, G_h^x) - A(u_h - P_h U_0, G_h^x)), \end{aligned}$$

which, together with (7.15), gives

$$\begin{aligned} \|U_0 - u_h\|_{1,\infty,\Omega} \leq C\|U_0 - P_h U_0\|_{1,\infty,\Omega} &+ |A_h(P_h U_0, G_h^x) - A(P_h U_0, G_h^x)|\\ &+ |A_h(u_h - P_h U_0, G_h^x) - A(u_h - P_h U_0, G_h^x)|. \end{aligned} \tag{7.16}$$

Applying (7.15) again and an inverse inequality, we have

$$|A_h(P_h U_0, G_h^x) - A(P_h U_0, G_h^x)| \leq C\overline{E}|\ln h|\|U_0\|_{2,\infty,\Omega}. \tag{7.17}$$

A similar argument yields

$$\begin{aligned} &|A_h(u_h - P_h U_0, G_h^x) - A(u_h - P_h U_0, G_h^x)|\\ &\quad \leq C\overline{E}|\ln h|\left(\|U_0 - u_h\|_{1,\infty,\Omega} + h\|U_0\|_{2,\infty,\Omega}\right). \end{aligned} \tag{7.18}$$

Substituting (7.17) and (7.18) into (7.16) implies that

$$\|U_0 - u_h\|_{1,\infty,\Omega} \leq C\left(h + \overline{E}|\ln h| + \overline{E}|\ln h|\|U_0 - u_h\|_{1,\infty,\Omega}\right). \tag{7.19}$$

If $\overline{E}|\ln h| < C_5 = 1/(2C)$, the desired result (7.12) follows from (7.19). ∎

Proof of Proposition 7.8. Let $m = h/(2\epsilon)$, and let χ_m^j be the solution of (7.3) on $[0, m]^d$, with the boundary condition $\chi_m^j = 0$ on $\partial[0, m]^d$. Set $\chi_m = (\chi_m^1, \chi_m^2, \ldots, \chi_m^d)^T$. u_ρ^m can be similarly defined using (7.5). Then it follows from the definition of $\overline{E}$ that

$$\overline{E} = |\mathcal{A} - [a(\mathcal{I} + \nabla\chi_m)]_m| \leq C(F_1 + F_2 + F_3), \tag{7.20}$$

where

$$\begin{aligned} F_1 &= |\mathcal{A} - [a(\mathcal{I} + \nabla u_\rho)]_m|, \\ F_2 &= |[a(\mathcal{I} + \nabla u_\rho) - a(\mathcal{I} + \nabla u_\rho^m)]_m|, \\ F_2 &= |[a\nabla(u_\rho^m - \chi_m)]_m|. \end{aligned}$$

Note that

$$F_1 = |\mathcal{A} - \langle a(\mathcal{I} + \nabla u_\rho)\rangle + [\varphi]_m|,$$

where $\varphi = \langle a(\mathcal{I} + \nabla u_\rho)\rangle - a(\mathcal{I} + \nabla u_\rho)$. It follows [21] that

$$\mathbb{E}|[\varphi]_m| \leq \sqrt{\mathbb{E}[\varphi]_m^2} \leq C\left(\frac{|\ln\rho^{-1}|^2}{\rho^{1/2}m}\right)^{d/2},$$

which, together with Lemma 7.5, gives

$$\mathbb{E}F_1 \leq C\left(G\rho^{(d-2-2\lambda)/(4+d)} + \left(\frac{|\ln\rho^{-1}|^2}{\rho^{1/2}m}\right)^{d/2}\right). \tag{7.21}$$

Let t_m be the first exit time of the domain $[0, 2m]^d$. Then $u_\rho^{2m} = M_x\Gamma(t_m)$, and for any $s > 0$,

$$\begin{aligned} |u_\rho - u_\rho^{2m}| &= \left|M_x\big(\Gamma(\infty) - \Gamma(t_m)\big)\right| \\ &\leq M_x\left\{|\Gamma(\infty)| + |\Gamma(t_m)| : t_m \leq s\right\} \\ &\quad + M_x\left\{e^{-s\rho}M_{\rho(s)}|\Gamma(\infty) - \Gamma(t_m)| : t_m > s\right\} \\ &\leq C\left(M_x\left((\Gamma(\infty))^2 + (\Gamma(t_m))^2\right)\right)^{1/2}\left(P_x\{t_m \leq s\}^{1/2} + e^{-s\rho}\right). \end{aligned}$$

Because $P_x\{t_m \leq s\} \leq e^{-Cm^2/s}$, we see that

$$\mathbb{E}\left[|u_\rho - u_\rho^{2m}|^2\right]_{2m} \leq \frac{CG^2}{\rho}\left(e^{-Cm^2/s} + e^{-s\rho}\right)^2,$$

whose optimization in s gives

$$\mathbb{E}\left[|u_\rho - u_\rho^{2m}|^2\right]_{2m} \leq \frac{CG^4}{\rho^2}e^{-Cm\rho^{1/2}}.$$

Now, it follows from standard interior estimates that

$$\begin{aligned} \mathbb{E}F_2 &\leq C\left(\mathbb{E}\left[|\nabla(u_\rho - u_\rho^m)|^2\right]_m\right)^{1/2} \leq \frac{C}{m}\left(\mathbb{E}\left[|u_\rho - u_\rho^{2m}|^2\right]_{2m}\right)^{1/2} \\ &\leq \frac{CG^2}{m\rho}e^{-Cm\rho^{1/2}}. \end{aligned} \tag{7.22}$$

Also, in an argument similar to that for F_1, we have

$$\mathbb{E}F_3 \le C\left(G^2\rho^{(d-2-2\lambda)/(4+d)} + \left(\frac{|\ln\rho^{-1}|^2}{\rho^{1/2}m}\right)^{d/2}\right). \tag{7.23}$$

Finally, substituting (7.21)–(7.23) into (7.20) generates

$$\mathbb{E}\overline{E} \le C\left(G^2\rho^{(d-2-2\lambda)/(4+d)} + \left(\frac{|\ln\rho^{-1}|^2}{\rho^{1/2}m}\right)^{d/2} + \frac{G^2}{m\rho}e^{-Cm\rho^{1/2}}\right).$$

Optimizing in ρ with respect to the first two terms of this inequality, we see that

$$\rho_o = m^{-(2d)/(d+4\beta)},$$

where $\beta = (d-2-2\lambda)/(d+4)$. Consequently, we obtain

$$\mathbb{E}\overline{E} \le C\left(\frac{|\ln m|^d}{m^\kappa} + \frac{G^2}{m\rho_0}e^{-Cm\rho_0^{1/2}}\right) \le C\frac{|\ln m|^d}{m^\kappa}, \tag{7.24}$$

where

$$\kappa = \frac{d/2}{1+\frac{d(d+4)/4}{d-2-2\lambda}}, \quad 0 < \lambda < \frac{1}{2}.$$

Absorbing the factor $|\ln m|^d$ into m^κ in (7.24) yields the desired result (7.13) for $d = 3$.

For $d = 1$, a direct evaluation gives

$$\mathbb{E}\overline{E}^2 \le \frac{C}{m},$$

where C is independent of m, which yields (7.13) with $\kappa = 1/2$. □

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