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On the heterogeneous multiscale method with various macroscopic solvers

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A B S T R A C T

The heterogeneous multiscale method (HMM) is a general method for efficient numerical solution of problems with multiscales. It consists of two components: an overall macroscopic solver for macrovariables on a macrogrid and an estimation of the missing macroscopic data from the microscopic model. In this paper we present a state-of-the-art review of the HMM with various macroscopic solvers, including finite differences, finite elements, discontinuous Galerkin, mixed finite elements, control volume finite elements, nonconforming finite elements, and mixed covolumes. The first four solvers have been studied in the HMM setting; the others are not. As example, the HMM with the nonconforming finite element macroscopic solver for nonlinear and random homogenization problems is also studied here.

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1. Introduction

E and Engquist introduced the heterogeneous multiscale method (HMM) for efficient numerical solution of problems with multiscales [1] (also see a recent review of HMM [2]). This multiscale method consists of two components: an overall macroscopic solver for macrovariables on a macrogrid and an estimation of the missing macroscopic data from the microscopic model. Numerical experiments and analysis have shown the potential of the HMM when the macroscopic solver uses finite difference and volume methods [3,1], finite element methods [4], mixed finite element methods [5], or discontinuous Galerkin methods [6]. The purpose of this paper is to present the HMM with these macroscopic solvers in a

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unified framework so that it also applies to other numerical discretization methods such as nonconforming, control volume, and mixed covolume finite element methods. These methods have not been studied in the HMM setting.

The finite difference HMM was studied in [3,1] from the original definition of finite differences. This method and the finite volume HMM can be treated in the context of the discontinuous Galerkin HMM when the trial and test functions are piecewise constants [6].

The analysis of the finite element HMM was given in detail in [4]. Here we generalize it to the control volume finite element HMM. The standard finite element method does not conserve mass locally, but the control volume finite element method does. With this generalization, the analysis also includes an application of the HMM to the finite element and finite difference box methods [7,8].

We extend the analysis of the mixed finite element HMM [5] to the mixed covolume HMM. The mixed covolume method has been developed in parallel to the mixed finite element method [9,10]. But it is often observed that the former produces smaller solution errors [11]. In addition, unlike the latter that uses a primary grid, the mixed covolume method uses a conservation law on a primary volume grid for a scalar variable and a constitutive law on a dual volume or covolume grid for a vector variable.

The discontinuous Galerkin HMM was studied in [6] for hyperbolic and parabolic problems. Here we develop and study it for elliptic homogenization problems as well. The discontinuous Galerkin method uses completely discontinuous piecewise polynomials. Recently, there has been increased interest in this method because of its localizability and parallelizability [12,13]. The nonconforming finite element method uses piecewise polynomial basis functions that are continuous at certain points on interelement edges or faces [12,14]. It has received considerable attention in solid and fluid mechanics because it involves much fewer degrees of freedom than the standard (conforming) finite element method. Here we propose and study the nonconforming finite element HMM. As example, in this paper we perform an error analysis in detail for this method for linear and nonlinear periodic and random homogenization problems.

The paper is organized as follows. In the next section we present the HMM in an abstract framework. Then the standard finite element, nonconforming finite element, control volume finite element, mixed finite element, mixed covolume, discontinuous Galerkin, and finite difference HMMs are, respectively, described in the third to ninth sections. The nonconforming finite element HMM for nonlinear and random homogenization problems is studied, respectively, in the tenth and eleventh sections.

We end with a remark that closely related multiscale methods, the multiscale (or mixed) finite element methods were developed in [15–17]. In particular, the multiscale finite element method (MsFEM) captures the effect of microscales on macroscales through modified bases [17] or through the modification of bilinear (quadratic) forms in the finite element formulation [5]. Its detailed analysis was carried out in [5,18–20]. As a general remark, the generic constant C is assumed to be independent of the mesh size H and the microscale ϵ throughout this paper. Finally, for numerical experiments of the HMM with different macroscopic solvers, the reader should refer to [21,22].

2. The HMM

2.1. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^d$, $1 \le d \le 3$, with Lipschitz boundary Γ. For a subdomain $D \subset \Omega$, each integer $m \ge 0$, and each real number $1 \le p \le \infty$, $W^{m,p}(D)$ indicates the usual Sobolev space of real functions that have all their weak derivatives of order up to m in the Lebesgue space $L^p(D)$. The norm and seminorm of $W^{m,p}(D)$ are denoted by $\|\cdot\|_{m,p,D}$ and $|\cdot|_{m,p,D}$, respectively. When $p = 2$, $W^{m,p}(D)$ is written as $H^m(D)$ with the norm $\|\cdot\|_{m,D}$ and the seminorm $|\cdot|_{m,D}$. We also use the space

$$H_0^1(D) = \{v \in H^1(D) : v|_{\partial D} = 0\}.$$

In the mixed finite element method, we will exploit the space

$$H(\mathrm{div}, D) = \left\{\tau \in (L^2(D))^d : \nabla \cdot \tau \in L^2(D)\right\}, \quad 1 \le d \le 3,$$

with the usual norm

$$\|\tau\|_{H(\mathrm{div},D)} = \left\{\|\tau\|_{0,D}^2 + \|\nabla \cdot \tau\|_{0,D}^2\right\}^{1/2}, \quad \tau \in H(\mathrm{div}, D).$$

For a rectangle D, we indicate by $C^\infty_{\mathrm{per}}(D)$ the set of C^∞ periodic functions with period D, and by $H^1_{\mathrm{per}}(D)$ the closure of $C^\infty_{\mathrm{per}}(D)$ under the H^1-norm. For a given space $H(D)$, we indicate by $\bar{H}(D)$ the subspace of $H(D)$ with zero integral mean over D.

In this and next seven sections, we consider the second-order elliptic problem

$$\begin{aligned} -\nabla \cdot \left(a_\epsilon \nabla u^\epsilon\right) &= f \quad \text{in } \Omega, \\ u^\epsilon &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{2.1}$$

where $f \in L^2(\Omega)$ is a given function and $a_\epsilon = \left(a_{ij}x, x/\epsilon\right)$ is a symmetric, positive definite, bounded tensor:

$$a_*|\eta|^2 \le \sum_{i,j=1}^d a_{ij}(x, y)\eta_i\eta_j \le a^*|\eta|^2 \quad \forall y,\ \eta \in \mathbb{R}^d, \tag{2.2}$$

for some positive constants a_* and a^*. In the next seven sections, we assume that $a(x, y)$ is smooth and periodic in y with period $Y = [-1/2, 1/2]^d$. In problem (2.1), the multiscale feature is reflected in the oscillatory nature of the coefficient a_ϵ for $\epsilon \ll 1$, which represents the microscale. For simplicity, we consider the homogeneous Dirichlet boundary condition in (2.1).

2.2. The HMM

A standard numerical method consists of discretizing the microscopic model (2.1) over the entire domain Ω. To capture the effect of the scales of interest, a large number of discretized equations are required if ϵ is small compared with the characteristic length of Ω. The primary idea of the HMM is to use both microscopic and macroscopic equations or models even if the latter are not explicitly known. Suppose that a microscopic process, such as quantum mechanics or molecular dynamics, describes the microscopic state variable u^ϵ, which is defined on a microscopic domain, and that a macroscopic process describes a macroscopic state variable U_0, which is defined on a macroscopic domain. The two processes and state variables are connected by the compression and reconstruction operators, Q and R: $Qu^\epsilon = U_0$, $RU_0 = u^\epsilon$, with the property $QR = I$, where I is the identity operator. For example, when the microscopic process is described by kinetic theory and the macroscopic process is described by hydrodynamics, the compression operator maps the one-particle phase-space distribution function to the conserved mass, momentum, and energy densities; the reconstruction operator does the opposite and is generally not unique [1]. The purpose is to approximate accurately the macroscopic state of the underlying system by using a macroscopic grid that resolves its large scale.

The HMM consists of two components: an overall macroscopic solver for the macroscopic state variable U_0 on a macroscopic grid and an estimation of the missing macroscopic data from the microscopic model. The macroscopic solver can be any reasonable numerical discretization method that will be concentrated on in this paper. If a macroscopic model with a coefficient A existed

$$-\nabla \cdot \left(A\nabla U_0\right) = f \quad \text{in } \Omega, \tag{2.3}$$

then an accurate approximation to the macroscopic variable U_0 would be sought through Eq. (2.3). That is, for $H > 0$, let T_H be a partition of Ω into elements $\{T\}$, and, associated with T_H, a macroscopic solver can be chosen to solve (2.3) for an approximation U_H:

$$\mathcal{L}(U_H) = F. \tag{2.4}$$

In the absence of an explicit representation such as (2.3), the missing macroscopic data for the macroscopic solver (2.4) must be estimated from the microscopic model (2.1). For each $T \in T_H$, let $\{x_l\}$ be certain points in T, and let u_l be some approximation to u^ϵ on $x_l + \epsilon Y$. Then we estimate the missing macroscopic data at x_l through some type of approximation scheme $\overline{\mathcal{L}}$:

$$\mathcal{L}(U_H) \approx \overline{\mathcal{L}}(u_l). \tag{2.5}$$

The approximation u_l is usually obtained by solving the original Eq. (2.1) on a small grid of $x_l+\epsilon Y$, which resolves the ϵ-scale. Because the number of the ϵ-cells $x_l + \epsilon Y$ is finite and the microscopic cell problems are independent, these problems can be solved in a parallel fashion. In this paper we will focus on the choice of various macroscopic solvers $\mathcal{L}$ and the estimation (2.5) of the missing macroscopic data for these solvers.

3. Standard finite element methods

The most obvious choice for the macroscopic solver $\mathcal{L}$ is the standard finite element method. The original finite element HMM was developed in [1], and its detailed analysis was given in [4]. If we had the macroscopic model (2.3), the next step in the finite element methods would be to evaluate the bilinear form

$$a(U_0, V) = (A\nabla U_0, \nabla V), \tag{3.1}$$

by some numerical quadrature:

$$a(U_0, V) \approx \sum_{T\in T_H} |T| \sum_{x_l\in T} \omega_l (A\nabla U_0) \cdot \nabla V(x_l), \tag{3.2}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^d$, as appropriate, $\omega_l > 0$ and x_l are the quadrature weights and points in T, and $|T|$ is the area or volume of T. In the absence of such an explicit expression, we must estimate the value of the integrand in (3.1) at these quadrature points.

To explain the idea of the finite element HMM, for $H > 0$, let T_H be a regular, quasi-uniform macroscale partition of Ω into triangles, where the mesh size H resolves the variations of Ω, f, and the slow variable of a_ϵ. Define

$$V_H = \{v \in H_0^1(\Omega) : v|_T \in P_r(T),\ T \in T_H\},$$

where $P_r(T)$ is the set of polynomials of degree at most $r \geq 1$ defined on triangle T.

For each $T \in T_H$ and $x_l \in T$, we denote by V_l the linear approximation of $V \in V_H$ at x_l; i.e.,

$$V_l(x) = V(x_l) + \nabla V(x_l) \cdot (x - x_l).$$

Also, set

$$\Delta_l = x_l + \epsilon Y.$$

Now, for any $V \in V_H$ we define $v_l^\epsilon \in V_l + \bar{H}^1_{\text{per}}(\Delta_l)$ by

$$\left(a_\epsilon \nabla v_l^\epsilon, \nabla w\right)_{\Delta_l} = 0 \qquad \forall w \in H^1_{\text{per}}(\Delta_l), \tag{3.3}$$

and approximate

$$(A\nabla U)\cdot\nabla V(x_l) \approx \frac{1}{|\Delta_l|}\left(a_\epsilon \nabla u_l^\epsilon, \nabla v_l^\epsilon\right)_{\Delta_l},$$

where u_l^ϵ corresponds to $U \in V_H$ through (3.3).

For any $U, V \in V_H$, we introduce the bilinear form

$$a_H(U, V) = \sum_{T\in T_H}\sum_{x_l\in T}\omega_l \frac{|T|}{|\Delta_l|}\left(a_\epsilon \nabla u_l^\epsilon, \nabla v_l^\epsilon\right)_{\Delta_l}.$$

Now, the finite element HMM is to seek $U_{\text{HMM}} \in V_H$ such that

$$a_H(U_{\text{HMM}}, V) = (f, V) \quad \forall V \in V_H. \tag{3.4}$$

Comparing the finite element HMM (3.4) with the standard finite element methods, one sees the modification of the bilinear form in the former, which needs the solution of local problems (3.3). It is through these local problems and the finite element formulation that the effect of microscales on macroscales can be correctly captured. As noted, because these local problems are independent of each other, they can be solved in parallel.

For the quadrature formula (3.2), we assume the rth-order numerical quadrature rule [23]:

$$\int_T v(x)\mathrm{d}x = |T|\sum_{l=1}^{L}\omega_l v(x_l) \quad \forall v \in P_{2r-2}(T),\ T \in T_H, \tag{3.5}$$

where $\omega_l > 0, l = 1, 2, \ldots, L$. For the case $r = 1$ we assume that this rule holds for $v \in P_1(T)$.

It can be seen [4] that (3.4) admits a unique solution, which satisfies

$$\|U_{\text{HMM}}\|_{1,\Omega} \le C\|f\|_{-1,\Omega}. \tag{3.6}$$

Moreover, if (3.5) holds and U_0 is sufficiently smooth, it can be proven [4] that

$$\begin{aligned} \|U_0 - U_{\text{HMM}}\|_{1,\Omega} &\le C(\epsilon + H^r), \\ \|U_0 - U_{\text{HMM}}\|_{0,\Omega} &\le C(\epsilon + H^{r+1}), \end{aligned} \tag{3.7}$$

where $U_0 \in H_0^1(\Omega)$ is the solution of the homogenized problem

$$(A\nabla U_0, \nabla V) = (f, V) \quad \forall V \in H_0^1(\Omega), \tag{3.8}$$

the homogenized matrix $A = (A_{ij})$ is given by

$$A_{ij}(x) = \frac{1}{|Y|}\int_Y \left(a_{ij}(x, y) + \sum_{k=1}^{d}\left(a_{ik}\frac{\partial \chi^j}{\partial y_k}\right)(x, y)\right)\mathrm{d}y, \tag{3.9}$$

and χ^j satisfies

$$-\nabla_y\cdot\left(a_\epsilon \nabla_y \chi^j(x, y)\right) = \sum_{i=1}^{d}\frac{\partial}{\partial y_i}a_{ij}(x, y), \quad y \in Y, \int_Y \chi^j(x, y)\mathrm{d}y = 0. \tag{3.10}$$

It is well known that A is symmetric and positive definite. We will assume that $\chi^j(x,\cdot) \in W^{1,\infty}(Y)$, which is true if $a_{ij}(x,\cdot) \in W^{1,\ell}(Y), \ell > 2$ [24].

It follows from (3.7) that the finite element HMM solution U_{HMM} provides a good approximation to the macroscopic solution U_0. We can introduce a simple reconstruction trick to retrieve the microscopic information from U_{HMM}. We define

$$u_H^\epsilon = U_{\text{HMM}} + \epsilon\sum_{k=1}^{d}\chi^k \frac{\partial U_{\text{HMM}}}{\partial x_k}. \tag{3.11}$$

Then it can be checked [4] that

$$\begin{aligned} \left(\sum_{T\in T_H}|u^\epsilon - u_H^\epsilon|^2_{1,T}\right)^{1/2} &\le C(\sqrt{\epsilon} + H^r), \\ \|u^\epsilon - u_H^\epsilon\|_{0,\Omega} &\le C(\epsilon + H^{r+1}). \end{aligned} \tag{3.12}$$

In the subsequent sections we will extend the standard finite element macroscopic solver to other solvers.

4. Nonconforming finite element methods

The most obvious extension for the macroscopic solver defined in the previous section is to the nonconforming finite element methods. The nonconforming finite element HMM has not been considered before. Compared with the standard (conforming) finite element methods analyzed in the previous section, finite element spaces used in the nonconforming methods employ fewer degrees of freedom, particularly for a fourth-order differential equation problem. To see the idea, however, throughout this section we will perform all proofs in detail for the lowest-order nonconforming finite element space on triangles (respectively, simplices) [12,25] for the second-order problem (2.1). We point out that there is no technical difficulty in extending all arguments to spaces of higher order and other types of nonconforming finite elements [26,12]. In the lowest-order case, the nonconforming finite element space V_H is

$$\begin{aligned} V_H = \{&V \in L^2(\Omega) : V|_T \in P_1(T),\ T \in T_H; V \text{ is continuous at the midpoints of interior} \\ &\text{edges (respectively, centroids of interior faces) and is zero at the} \\ &\text{midpoints of edges (respectively, centroids of faces) on } \Gamma\}. \end{aligned}$$

In the present case, if we had the macroscopic model (2.3), we would evaluate the bilinear form

$$a_H(U, V) = \sum_{T\in T_H} (A\nabla U, \nabla V)_T,$$

by the numerical quadrature

$$a_H(U, V) \approx \sum_{T\in T_H} |T|(A\nabla U)\cdot \nabla V(x_T),$$

where x_T is the barycenter of $T \in T_H$. In the absence of an explicit expression for A, we must estimate the missing data at x_T, which are obtained by solving a microscale problem on a microscopic cell:

$$\Delta_T = x_T + \epsilon Y.$$

For any $V \in V_H$, we define $v_T^\epsilon \in V + \bar{H}^1_{\text{per}}(\Delta_T)$ by

$$\left(a(x_T, x/\epsilon)\nabla v_T^\epsilon, \nabla w\right)_{\Delta_T} = 0 \quad \forall w \in H^1_{\text{per}}(\Delta_T). \tag{4.1}$$

Furthermore, for any $U,\ V \in V_H$, we introduce the bilinear form

$$a_H(U, V) = \sum_{T\in T_H} \frac{|T|}{|\Delta_T|}\left(a_\epsilon \nabla u_T^\epsilon, \nabla v_T^\epsilon\right)_{\Delta_T},$$

where u_T^ϵ corresponds to $U \in V_H$ through (4.1). Then the nonconforming finite element HMM can be defined as in (3.4). In addition, existence and uniqueness of the solution $U_{\text{HMM}} \in V_H$ can be shown as in the conforming case. In particular, the following coercivity and continuity of $a_H(\cdot,\cdot)$ can be shown using assumption (2.2) and Eq. (4.1):

$$\begin{aligned} &a_H(V, V) \geq C_1|V|_H^2 \quad \forall V \in V_H, \\ &a_H(U, V) \leq C_2|U|_H|V|_H \quad \forall U,\ V \in V_H, \end{aligned} \tag{4.2}$$

where the constants C_1 and C_2 are independent of H and $|V|_H = \left(\sum_{T\in T_H}\|\nabla V\|_{0,T}^2\right)^{1/2}$. It follows from (4.2) that the stability result (3.6) holds as well. Below we focus on an error analysis of the nonconforming finite element HMM.

For the nonconforming finite element methods, it is known that the Céa lemma is no longer valid. The next Strang's second lemma [27] can be easily shown using (4.2) [12,14].

Lemma 4.1. *Let U_0 and U_{HMM} be the respective solutions of (3.8) and the nonconforming HMM solution. Then there is a constant $C > 0$, independent of H and ϵ, such that*

$$|\Pi_H U_0 - U_{\text{HMM}}|_H \leq C\left\{\inf_{V\in V_H}|\Pi_H U_0 - V|_H + \sup_{W\in V_H, W\neq 0}\frac{|a_H(\Pi_H U_0, W) - (f, W)|}{|W|_H}\right\}, \tag{4.3}$$

where Π_H is the standard interpolation operator into V_H.

In (4.3), the first term on the right-hand side is referred to as the approximation error, and the second term is called the consistency error. The latter error stems from nonconformity.

Theorem 4.2. *Let U_0 and U_{HMM} be the respective solutions of (3.8) and the nonconforming HMM solution. Then there is a constant $C > 0$, independent of H and ϵ, such that*

$$\begin{aligned} &|U_0 - U_{\text{HMM}}|_H \leq C(\epsilon + H)\|f\|_{0,\Omega}, \\ &\|U_0 - U_{\text{HMM}}\|_{0,\Omega} \leq C(\epsilon + H^2)\|f\|_{0,\Omega}. \end{aligned} \tag{4.4}$$

Proof. Because the conforming P_1 finite element space is a subspace of V_H, we see that

$$\inf_{V\in V_H} |\Pi_H U_0 - V|_H \le |\Pi_H U_0 - U_0|_H + \inf_{V\in V_H} |U_0 - V|_H \le CH|U_0|_{2,\Omega}, \tag{4.5}$$

so it suffices to bound the consistency error in (4.3).

It follows from (3.10) that the solution of the cell problem (4.1) has the explicit expression

$$v_T^\epsilon = V + \epsilon \sum_{j=1}^{d} \chi^j(x_T, x/\epsilon)\frac{\partial V}{\partial x_j}, \tag{4.6}$$

which, together with (3.9), implies that

$$\frac{1}{|\Delta_T|}\int_{\Delta_T} a(x_T, x/\epsilon)\nabla v_T^\epsilon = A(x_T)\nabla V, \quad V \in V_H. \tag{4.7}$$

Consequently, we see that

$$\frac{|T|}{|\Delta_T|}\left(a(x_T, x/\epsilon)\nabla u_T^\epsilon, \nabla v_T^\epsilon\right)_{\Delta_T} = (A(x_T)\nabla U, \nabla V)_T, \quad U,\ V \in V_H,\ T \in T_H. \tag{4.8}$$

Let u_0^ϵ correspond to $\Pi_H U_0$ through (4.1). Then, using (4.8), we write

$$\begin{aligned} a_H(\Pi_H U_0, W) - (f, W) &= \sum_{T\in T_H} \frac{|T|}{|\Delta_T|}\left(a_\epsilon \nabla u_0^\epsilon, \nabla w_T^\epsilon\right)_{\Delta_T} - (f, W) \\ &= \sum_{T\in T_H} \frac{|T|}{|\Delta_T|}\left(\{a(x, x/\epsilon) - a(x_T, x/\epsilon)\}\nabla u_0^\epsilon, \nabla w_T^\epsilon\right)_{\Delta_T} \\ &\quad + \sum_{T\in T_H} \{(A(x_T)\nabla \Pi_H U_0, \nabla W)_T - (A(x)\nabla U_0, \nabla W)_T\} \\ &\quad + \sum_{T\in T_H} (A(x)\nabla U_0, \nabla W)_T - (f, W). \end{aligned} \tag{4.9}$$

Each of the terms in (4.9) can be estimated as follows. First, we have

$$\left|\sum_{T\in T_H} \frac{|T|}{|\Delta_T|}\left(\{a(x, x/\epsilon) - a(x_T, x/\epsilon)\}\nabla u_0^\epsilon, \nabla w_T^\epsilon\right)_{\Delta_T}\right| \le C\epsilon \sum_{T\in T_H}\frac{|T|}{|\Delta_T|}|\Pi_H U_0|_{1,\Delta_T}|W|_{1,\Delta_T} \le C\epsilon|U_0|_{1,\Omega}|W|_H.$$

Second, we see that

$$\left|\sum_{T\in T_H} \{(A(x_T)\nabla \Pi_H U_0, \nabla W)_T - (A(x)\nabla U_0, \nabla W)_T\}\right| \le C\,(\epsilon + H)\,\|U_0\|_{2,\Omega}\|W\|_H.$$

Third, application of the standard convergence argument for the nonconforming finite element method under consideration [12] yields

$$\left|\sum_{T\in T_H} (A(x)\nabla U_0, \nabla W)_T - (f, W)\right| \le CH|U_0|_{2,\Omega}\|W\|_H.$$

Substituting these three inequalities into (4.9), we see that

$$|a_H(\Pi_H U_0, W) - (f, W)| \le C\,(\epsilon + H)\,\|U_0\|_{2,\Omega}\|W\|_H,$$

which, together with (4.3) and (4.5), gives the first inequality in (4.4). The second inequality can be shown using a standard duality argument for the nonconforming finite element methods [12]. □

The microscopic information can be retrieved as in the conforming case. That is, with u_H^ϵ defined as in (3.11), it can be shown that

$$\begin{aligned} &|u^\epsilon - u_H^\epsilon|_H \le C(\sqrt{\epsilon} + H)\|f\|_{0,\Omega} \\ &\|u^\epsilon - u_H^\epsilon\|_{0,\Omega} \le C(\epsilon + H^2)\|f\|_{0,\Omega}. \end{aligned}$$

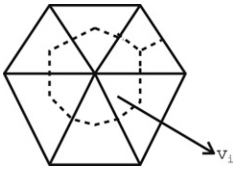

Fig. 1. A control volume V_i (dashed).

5. Control volume finite element methods

Another closely related class of discretization methods are the control volume finite element (CVFE) methods for solution of partial differential equations. Control volumes are constructed from base elements such as triangles or rectangles (see Fig. 1), and the CVFE methods are then defined in terms of an integral formulation of the differential equations on the boundaries of these volumes [28]. These methods are also referred to as the box methods [7] or as finite volume element methods [29]. Regardless of their physical interpretations, the CVFE methods can be mathematically treated as Petrov–Galerkin methods with trial function spaces associated with certain finite element spaces and test spaces related to control volumes. They lie somewhere between the finite element and finite difference methods. It is well known that the standard finite element methods do not conserve mass locally (at the element level). The CVFE methods conserve mass locally on control volumes.

The CVFE methods can be chosen as the macroscopic solver in the HMM for solution of multiscale problems. However, because they are usually implemented in terms of the standard finite element methods by using an equivalence relation between these two methods [7,28], we only need to estimate the missing macroscopic data at the quadrature points of a triangle or rectangle, which can be carried out exactly in the same way as in the third section. Thus we omit the development of the CVFE-HMM.

6. Mixed finite element methods

In this section we extend the finite element HMM to the mixed finite element methods. The mixed finite element HMM was analyzed in detail in [5]. In the porous media flow application, for example, u^ϵ in (2.1) represents a pressure, and the variable $q^\epsilon = -a_\epsilon \nabla u^\epsilon$ represents a fluid velocity. The reason for using the mixed methods is, among others, that the variable q^ϵ is the primary variable in which one is interested. Then the mixed methods are developed to approximate both u^ϵ and q^ϵ simultaneously and to give a high-order approximation of both variables.

Again, for $H > 0$, let T_H be a regular, quasi-uniform macroscale partition of Ω. Associated with T_H, let $V_H \subset V = H(\mathrm{div}, \Omega)$ and $W_H \subset L^2(\Omega)$ be the classical mixed finite element Raviart–Thomas (if $d = 2$) [30], Nedelec (if $d = 3$) [31], Brezzi–Douglas–Fortin–Marini [32], Brezzi–Douglas–Marini (if $d = 2$) [33], Brezzi–Douglas–Durán–Fortin (if $d = 3$) [34], or Chen–Douglas [35] spaces for second-order partial differential equations. These spaces satisfy the *inf-sup* condition

$$\sup_{\tau \in V_H, \tau \neq 0} \frac{(\nabla \cdot \tau, V)}{\|\tau\|_{H(\mathrm{div},\Omega)}} \geq C\|V\|_{0,\Omega} \quad \forall V \in W_H.$$

For each $T \in T_H$, they also have the approximation properties

$$\begin{aligned} &\inf_{\tau_H \in V_H(T)} \|\tau - \tau_H\|_{0,T} \leq CH_T^l \|\tau\|_{l,T}, \quad 1 \leq l \leq r+1, \\ &\inf_{\tau_H \in V_H(T)} \|\nabla \cdot (\tau - \tau_H)\|_{0,T} \leq CH_T^l \|\nabla \cdot \tau\|_{l,T}, \quad 0 \leq l \leq r^*, \\ &\inf_{V_H \in W_H(T)} \|V - V_H\|_{0,T} \leq CH_T^l \|V\|_{l,T}, \quad 0 \leq l \leq r^*, \end{aligned} \tag{6.1}$$

where $r^* = r+1$ for the Raviart–Thomas, Nedelec, Brezzi–Douglas–Fortin–Marini, and first and third Chen–Douglas spaces, $r^* = r > 0$ for the Brezzi–Douglas–Marini, Brezzi–Douglas-Durán–Fortin, and second Chen–Douglas spaces, $V_H(T) = V_H|_T$, $W_H(T) = W_H|_T$, and $H_T = \mathrm{diam}(T)$.

As for the standard finite element HMM, we assume the accuracy condition for a numerical quadrature formula: For a given mixed space V_H,

$$\int_T (\sigma \cdot \tau)(x)\mathrm{d}x = |T| \sum_{l=1}^{L} \omega_l (\sigma \cdot \tau)(x_l), \quad \sigma, \tau \in V_H,\ T \in T_H, \tag{6.2}$$

where $\omega_l > 0$ and $x_l \in T$ are quadrature weights and points, $l = 1, 2, \ldots, L$. It is at these quadrature points where there may be no explicit knowledge of A. Thus, for each x_l, we set

$$\Delta_l = x_l + \epsilon Y, \quad l = 1, 2, \ldots, L,$$

and, for any $\tau \in V_H$, we define $v_l \in \bar{H}^1(\Delta_l)$ by

$$(a_\epsilon \nabla v_l, \nabla w)_{\Delta_l} = -(\tau(x_l), \nabla w)_{\Delta_l} \quad \forall w \in H^1(\Delta_l),\ T \in T_H,\ l = 1, 2, \ldots, L, \tag{6.3}$$

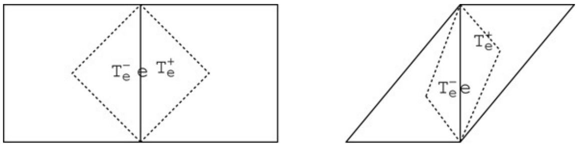

Fig. 2. Primal and dual grids: $T'_e = T_e^- \cup e \cup T_e^+$.

and

$$V_l = -a_\epsilon \nabla v_l, \quad x \in \Delta_l,\ l = 1, 2, \dots, L. \tag{6.4}$$

Now, for any $\sigma, \tau \in V_H$, we define the bilinear form

$$a_H(\sigma, \tau) = \sum_{T \in T_H} \sum_{l=1}^{L} \frac{|T|}{|\Delta_l|} \omega_l \left(a_\epsilon^{-1} U_l, V_l\right)_{\Delta_l}, \tag{6.5}$$

where U_l corresponds to $\sigma \in V_H$ through (6.3) and (6.4).

The mixed finite element HMM for (2.1) is defined: Find $q_{\mathrm{HMM}} \in V_H$ and $U_{\mathrm{HMM}} \in W_H$ such that

$$\begin{aligned} &a_H(q_{\mathrm{HMM}}, \tau) - (\nabla \cdot \tau, U_{\mathrm{HMM}}) = 0 \quad \forall \tau \in V_H, \\ &(\nabla \cdot q_{\mathrm{HMM}}, V) = (f, V) \quad \forall V \in W_H. \end{aligned} \tag{6.6}$$

Under the quadrature formula (6.2), problem (6.6) has a unique solution $q_{\mathrm{HMM}} \in V_H$ and $U_{\mathrm{HMM}} \in W_H$, which satisfies [5]

$$\|q_{\mathrm{HMM}}\|_{H(\mathrm{div},\Omega)} + \|U_{\mathrm{HMM}}\|_{0,\Omega} \le C \|f\|_{0,\Omega}.$$

Furthermore, if U_0 and $Q_0 = -A\nabla U_0$ are sufficiently smooth, the error estimates hold

$$\begin{aligned} &\|Q_0 - q_{\mathrm{HMM}}\|_{0,\Omega} \le C\left(H^{r+1} + \epsilon\right), \\ &\|U_0 - U_{\mathrm{HMM}}\|_{0,\Omega} \le C\left(H^{r^*} + H^{r+1} + \epsilon\right), \\ &\|\nabla \cdot (Q_0 - q_{\mathrm{HMM}})\|_{0,\Omega} \le CH^{r^*}, \end{aligned} \tag{6.7}$$

where r and r^* are defined as in (6.1).

From (6.7) we see that the mixed finite element HMM solutions U_{HMM} and q_{HMM} provide a good approximation to the macroscopic solution U_0 and Q_0, respectively. As for the standard and nonconforming finite element HMMs, we can use U_{HMM} and q_{HMM} to retrieve the microscopic information. For more details, please refer to [5].

7. Mixed covolume methods

As noted in the previous section, the standard mixed finite element methods are introduced to approximate both u^ϵ and q^ϵ simultaneously and to give a high-order approximation of both variables. On the other hand, mixed covolume methods have been developed with the same purpose, in addition to that they often produce smaller solution errors [11]. Unlike the standard mixed methods that use a primary grid T_H, the mixed covolume methods use a conservation law on a primary volume grid T_H for the scalar variable and a constitutive law on a dual volume or covolume grid T'_H for the vector variable. Depending on how they are interpreted, these methods are referred to as mixed covolume methods (preferred by us), mixed control volume methods, and mixed balance methods [36–39,10]. Regardless of their physical interpretations, this class of numerical methods can also mathematically be studied as Petrov–Galerkin methods with trial spaces associated with certain finite element spaces and test spaces related to finite volumes, as in the CVFE methods.

The most well-known example for the dual grid T'_H is the MAC (marker and cell) method that employs two staggered rectangular grids [40]. These covolume methods can use either nonoverlapping (see Fig. 2) or overlapping (see Fig. 3) covolumes. The left-hand side figure in Fig. 2 is a primary partition that consists of rectangles, and a typical interior covolume in its dual partition is the dashed quadrilateral, the union of two triangles $T_e^- \cup T_e^+$ with the common edge e in T_H. The two vertices inside the two rectangles are their centers. Note that each edge in T_H corresponds to a covolume. Near the boundary Γ a covolume is either T_e^- or T_e^+. The right-hand side figure in Fig. 2 has an analogous meaning when the primary partition T_H consists of triangles. On the other hand, the dashed covolumes in Fig. 3 are overlapping.

Based on the earlier results of Chou and Kwak [9,37], a unified framework was presented for a number of mixed covolume methods [41]. This framework connects all these methods to the standard mixed finite element methods using an injective mapping γ_H from the space V_H (see the previous section) to a test space $\mathcal{Y}_H$. In this section we define the mixed covolume HHH using this mapping.

The definitions (6.3) and (6.4) remain valid. But for any $\sigma, \tau \in V_H$, the bilinear form $a_H(\cdot,\cdot)$ is modified to

$$a_H(\sigma, \tau) = \sum_{T \in T_H} \sum_{l=1}^{L} \frac{|T|}{|\Delta_l|} \omega_l \left(a_\epsilon^{-1} U_l, V'_l\right)_{\Delta_l}, \tag{7.1}$$

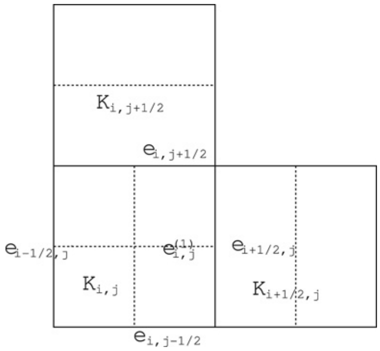

Fig. 3. Dual grid with overlapping covolumes.

where U_l and V_l', respectively, correspond to σ and $\gamma_H \tau$ through (6.3) and (6.4). With this modification, the mixed covolume HHH for (2.1) can be defined in the same manner as in (6.6).

To have the well-posedness of system (6.6) for the mixed covolume HHH, two hypotheses are required for the mapping γ_H. We assume that there are positive constants C such that

$$\begin{aligned} &(a_\epsilon^{-1}\tau, \gamma_H \tau) \geq C\|\tau\|_{0,\Omega}^2 \quad \forall \tau \in V_H, \\ &\|\gamma_H \tau\|_{0,\Omega} \leq C\|\tau\|_{0,\Omega} \quad \forall \tau \in V_H. \end{aligned} \tag{7.2}$$

Furthermore, to establish error estimates, we need an approximation property of γ_H. For example, in the case of the lowest-order Raviart–Thomas mixed finite element spaces (i.e., $r = 0$ in (6.1)), we assume that

$$|(\sigma, \tau - \gamma_H \tau)| \leq CH\|\sigma\|_{1,\Omega}\|\tau\|_{0,\Omega}, \quad \sigma \in H^1(\Omega),\ \tau \in V_H. \tag{7.3}$$

Examples of γ_H that satisfy both (7.2) and (7.3) can be found in [42,43]. With these two assumptions, the mixed covolume HHH can be analyzed as for the standard mixed finite element HHH [5]. Moreover, under (7.3), the error estimates hold:

$$\begin{aligned} &\|Q_0 - q_{\text{HMM}}\|_{0,\Omega} \leq C\left(H + \epsilon\right), \\ &\|U_0 - U_{\text{HMM}}\|_{0,\Omega} \leq C\left(H + \epsilon\right), \\ &\|\nabla \cdot (Q_0 - q_{\text{HMM}})\|_{0,\Omega} \leq CH. \end{aligned} \tag{7.4}$$

8. Discontinuous Galerkin methods

In this section we consider the case where the macroscopic solver $\mathcal{L}$ is the discontinuous Galerkin (DG) method. Unlike the standard finite element methods, for a given diffusion problem such as (2.1), the DG methods can have many different formulations that result in different methods. Essentially, there are three DG methods: symmetric, nonsymmetric, and mixed (Local) DG. For a pure diffusion problem, these methods may not be stable, so various stabilized (penalty) DG methods have been proposed. For a collection of these methods and their comparison, the reader may refer to [12]. Recently, Chen, E, and Shu has studied a LDG-HMM for one-dimensional hyperbolic and parabolic multiscale problems [6] that will be reviewed here. To be consistent with (2.1), instead of working with a time-dependent problem, we consider the diffusion–reaction problem

$$\begin{aligned} &-\left(a_\epsilon u_x^\epsilon\right)_x + u^\epsilon = f \quad \text{in } [0, 2\pi], \\ &u^\epsilon(0) = u^\epsilon(2\pi), \end{aligned} \tag{8.1}$$

where $a_\epsilon(x) = a\left(x, x/\epsilon\right) > 0$, with $a(x, y)$ smooth in x and periodic in y with the period $Y = [0, 2\pi]$. The reason to include a reaction term is for the subsequent LDG method to have a unique solution. This term may be thought of as arising from the discretization of a time differentiation term.

We denote a regular grid by $Y_j = [x_{j-1/2}, x_{j+1/2}]$, $j = 1, 2, \ldots, N$, with the cell center $x_j = (x_{j-1/2} + x_{j+1/2})/2$ and the cell size $H_j = x_{j+1/2} - x_{j-1/2}$. Set $H = \max_j H_j$, and introduce the finite element space

$$V_H = \{V \in L^2(\Omega) : V|_{Y_j} \in P_r(Y_j),\ j = 1, 2, \ldots, N\}, \quad r \geq 0.$$

For $U, W \in V_H$, on the microcell $Y_j^\epsilon = [x_{j+1/2}, x_{j+1/2} + 2\pi\epsilon]$ we solve the problem

$$\begin{aligned} &-\left(a_\epsilon \hat{u}_x^\epsilon\right)_x + \hat{u}^\epsilon = 0 \quad \text{in } Y_j^\epsilon, \\ &\hat{u}^\epsilon - \left(U_{j+1/2}^+ + W_{j+1/2}^+(x - x_{j+1/2})\right) \quad \text{is periodic,} \end{aligned} \tag{8.2}$$

where $U^+_{j+1/2}$ and $U^-_{j+1/2}$ (below) indicate the right and left limits of U at the grid points $x_{j+1/2}$. Then we define

$$(F_\epsilon(U, W, x))^+_{j+1/2} = \frac{1}{2\pi\epsilon}\int_{Y_j} a_\epsilon \hat{u}^\epsilon_x \mathrm{d}x.$$

Now, the LDG-HMM for (8.1) is defined: Find $U = U_{\mathrm{HMM}}, W = W_{\mathrm{HMM}} \in V_H$ such that

$$\begin{aligned} &(W, v)_{Y_j} + (U, v)_{Y_j} - U^-_{j+1/2} v^-_{j+1/2} + U^-_{j-1/2} v^+_{j-1/2} = 0 \quad \forall v \in V_H, \\ &(a_\epsilon W, w)_{Y_j} + (U, w)_{Y_j} - F^+_{j+1/2} w^-_{j+1/2} + F^+_{j-1/2} w^+_{j-1/2} = (f, w)_{Y_j} \quad \forall w \in V_H, \end{aligned} \tag{8.3}$$

for $j = 1, 2, \ldots, N$, where $F^+_{j+1/2} = (F_\epsilon(U, W, x))^+_{j+1/2}$.

Let U_0 be the exact solution of the homogenized problem of (8.1). If U_0 and $Q_0 = (U_0)_x$ are sufficiently smooth, it can be shown [6] that

$$\|U_0 - U_{\mathrm{HMM}}\|_{0,\Omega} + \|Q_0 - W_{\mathrm{HMM}}\|_{0,\Omega} \le C\left(\frac{\epsilon}{H} + H^{r+1}\right), \quad r \ge 0. \tag{8.4}$$

Note that this estimate is optimal in terms of H. However, compared with the error estimates obtained in the previous sections, it is not sharp with respect to ϵ. Future research will be needed to improve this estimate.

9. Finite difference methods

In the previous sections we have chosen various finite element method as the macroscopic solver $\mathcal{L}$ in the HMM. For completeness, we now consider the finite difference method as the macroscopic solver that was developed in [3,1] for a time-dependent parabolic multiscale problem. Again, to be consistent with the multiscale problem under consideration, we stick with problem (2.1). Also, a point-distributed finite difference method was used in [3,1], while a block-centered difference is utilized here, which is consistent with the DG methods considered in the previous section.

Let $\Omega = [0, 1]^2$ be the unit square, and denote a partition of Ω into smaller squares of equal size by $Y_{ij} = [x_{1,i-1/2}, x_{1,i+1/2}] \times [x_{2,j-1/2}, x_{2,j+1/2}]$, $i, j = 1, 2, \ldots, N$, with the cell center $(x_{1,i}, x_{2,j})$, where $x_{1,i} = (x_{1,i-1/2} + x_{1,i+1/2})/2$ and $x_{2,j} = (x_{2,j-1/2} + x_{2,j+1/2})/2$. Corresponding to this partition, we define the finite element space (i.e., the space of piecewise constants)

$$V_H = \{V \in L^2(\Omega) : V|_{Y_{ij}} \in P_0(Y_{ij}),\ i, j = 1, 2, \ldots, N\}.$$

Also, associated with each point $(x_{1,i+1/2}, x_{2,j})$, we introduce a microscopic ϵ-cell:

$$Y^\epsilon_{i+1/2,j} = \left[x_{1,i+1/2} - \frac{\epsilon}{2}, x_{1,i+1/2} + \frac{\epsilon}{2}\right] \times \left[x_{2,j} - \frac{\epsilon}{2}, x_{2,j} + \frac{\epsilon}{2}\right], \quad i = 0, 1, \ldots, N,\ j = 1, 2, \ldots, N.$$

Similarly, $Y^\epsilon_{i,j+1/2}$ can be defined. Furthermore, for each $V \in V_H$ we define its linear approximation on $Y^\epsilon_{i+1/2,j}$

$$V_{i+1/2,j}(x_1, x_2) = V_{ij} + \frac{V_{i+1,j} - V_{ij}}{H} x_1, \quad (x_1, x_2) \in Y^\epsilon_{i+1/2,j},$$

where $V_{ij} = V|_{Y_{ij}}$. $V_{i+1/2,j}(x_1, x_2)$ can be defined in an analogous manner.

Now, for any $V \in V_H$, on each $Y^\epsilon_{i+1/2,j}$ we define $v^\epsilon_{i+1/2,j} \in V_{i+1/2,j} + \bar{H}^1_{\mathrm{per}}(Y^\epsilon_{i+1/2,j})$ by

$$\left(a_\epsilon \nabla v^\epsilon_{i+1/2,j}, \nabla w\right)_{Y^\epsilon_{i+1/2,j}} = 0 \quad \forall w \in H^1_{\mathrm{per}}(Y^\epsilon_{i+1/2,j}), \tag{9.1}$$

and

$$P_{i+1/2,j}(V) = -\frac{1}{|Y^\epsilon_{i+1/2,j}|}\int_{Y^\epsilon_{i+1/2,j}} a_\epsilon \nabla v^\epsilon_{i+1/2,j} \mathrm{d}x, \quad i = 0, 1, \ldots, N,\ \ j = 1, 2, \ldots, N. \tag{9.2}$$

The vector $P_{i,j+1/2}(V)$ can be defined similarly. Finally, the FD-HMM is defined: Find $U_{\mathrm{HMM}} \in V_H$ such that, for $i, j = 1, 2, \ldots, N$,

$$\frac{P_{1,i+1/2,j}(U_{\mathrm{HMM}}) - P_{1,i-1/2,j}(U_{\mathrm{HMM}}) + P_{2,i,j+1/2}(U_{\mathrm{HMM}}) - P_{2,i,j+1/2}(U_{\mathrm{HMM}})}{H} = f_{ij}, \tag{9.3}$$

where $P_{i+1/2,j} = (P_{1,i+1/2,j}, P_{2,i+1/2,j})$.

If U_0 is the exact solution of the homogenized problem of (2.1), it can be proven [3] that

$$\|U_0 - U_{\mathrm{HMM}}\|_{0,\Omega} \le C\left(\frac{\epsilon}{H} + H\right). \tag{9.4}$$

As in the LCD-HMM, estimate (9.4) is not sharp with respect to ϵ.

10. A nonlinear problem

The HMM with the standard finite element methods as the macroscopic solver has been analyzed for nonlinear homogenization problems [4]. As an example of the possible extensions to other macroscopic solvers presented in the previous sections, we study the nonconforming finite element HMM. The notation given in the fourth section applies, and the argument here follows [4].

We consider the nonlinear problem

$$\begin{aligned} -\nabla\cdot\left(a_\epsilon \nabla u^\epsilon\right) &= f \quad \text{in } \Omega, \\ u^\epsilon &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{10.1}$$

where $a_\epsilon = a(x, x/\epsilon, u^\epsilon)$ now depends on the solution u^ϵ. We assume that the coefficient $a(x, y, z)$ is equi-continuous in z uniformly with respect to x and y and periodic in y with period $Y = [-1/2, 1/2]^d$. Furthermore, it satisfies inequality (2.2). Under such assumptions, the solution u^ϵ converges weakly in $\mathcal{U} = W_0^{1,p}(\Omega)$ $(p > 1)$ to the solution of the homogenized equation [44]

$$\begin{aligned} -\nabla\cdot\left(A(x, U_0)\nabla U_0\right) &= f \quad \text{in } \Omega, \\ U_0 &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{10.2}$$

where the homogenized matrix $A = (A_{ij})$ is

$$A_{ij}(x, q) = \frac{1}{|Y|}\int_Y \left(a_{ij}(x, y, q) + \sum_{k=1}^d \left(a_{ik}\frac{\partial \chi^j}{\partial y_k}\right)(x, y, q)\right) \mathrm{d}y \quad \forall q \in \mathbb{R},$$

and χ^j satisfies, with a periodic boundary condition in y,

$$\begin{aligned} -\nabla_y\cdot\left(a_\epsilon(x, y, q)\nabla_y \chi^j\right) &= \sum_{i=1}^d \frac{\partial}{\partial y_i} a_{ij}(x, y, q), \quad y \in Y, \\ \int_Y \chi^j(x, y, q)\mathrm{d}y &= 0, \quad q \in \mathbb{R}. \end{aligned} \tag{10.3}$$

As in the linear case (3.8), the variational form of (10.2) reads: Find $U_0 \in \mathcal{U}$ such that

$$(A(x, U_0)\nabla U_0, \nabla v) = (f, v) \quad \forall v \in \mathcal{U}. \tag{10.4}$$

Let $V_H \subset L^2(\Omega)$ be the nonconforming finite element space defined in the fourth section. For any $V \in V_H$, we define $v_T^\epsilon \in V + H^1_{\text{per}}(\Delta_T)$ by

$$\left(a(x_T, x/\epsilon, v_T^\epsilon)\nabla v_T^\epsilon, \nabla w\right)_{\Delta_T} = 0 \quad \forall w \in H^1_{\text{per}}(\Delta_T). \tag{10.5}$$

Furthermore, for any $U, V \in V_H$, we introduce the bilinear form

$$a_H(U, V) = \sum_{T\in T_H} \frac{|T|}{|\Delta_T|}\left(a(x_T, x/\epsilon, u_T^\epsilon)\nabla u_T^\epsilon, \nabla v_T^\epsilon\right)_{\Delta_T},$$

where u_T^ϵ corresponds to $U \in V_H$ through (10.5). Now, as in the linear case, the nonconforming finite element HMM for (10.1) is to seek $U_{\text{HMM}} \in V_H$ such that

$$a_H(U_{\text{HMM}}, V) = (f, V) \quad \forall V \in V_H. \tag{10.6}$$

To abuse the notation, whenever bilinear (quadratic) forms and norms involving partial derivatives are evaluated on the nonconforming finite element space V_H, they are understood in the piecewise sense, as in the definition of the norm $|\cdot|_H$. Introduce the linearized differential operator at U_0:

$$L_1(U_0)v = -\nabla\cdot\left(A(x, U_0)\nabla v + v\,A_p(x, U_0)\nabla U_0\right), \quad v \in H^1(\Omega),$$

and the corresponding bilinear form

$$\hat{a}(U_0; v, w) = (A(x, U_0)\nabla v, \nabla w) + \left(vA_p(x, U_0)\nabla U_0, \nabla w\right) \quad \forall v, \ w \in H^1(\Omega),$$

where $A_p(x, u) = \nabla_u A(x, u)$. We assume that this linearized operator is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, so U_0 is an isolated solution of (10.4). Moreover, application of an argument in [45] to the nonconforming finite element method considered implies that there is $H_0 > 0$ such that for $0 < H < H_0$ [45],

$$\sup_{W\in V_H, W\neq 0} \frac{\hat{a}(U_0; V, W)}{\|W\|_{1,\Omega}} \geq C_0\|V\|_{1,\Omega} \quad \forall V \in V_H, \tag{10.7}$$

where $C_0 > 0$ is independent of H.

For any $v, v_1, w \in \mathcal{U}$, we define

$$\mathcal{R}(v, v_1, w) = a(v_1, w) - a(v, w) - \hat{a}(v; v_1 - v, w),$$

where $a(v, w) = (A(x, v)\nabla v, \nabla w)$. If $v, v_1 \in \mathcal{U}$ satisfy $\|v\|_{1,\infty,\Omega} + \|v_1\|_{1,\infty,\Omega} \le M$, then extension of a technique in [46] to the nonconforming method gives

$$|\mathcal{R}(v, v_1, w)| \le C(M)\left(\|e\|_{0,2p}^2 + \|e\nabla e\|_{0,p}\right)\|\nabla w\|_{0,q}, \quad e = v - v_1, \ \frac{1}{p} + \frac{1}{q} = 1. \tag{10.8}$$

It now follows from the definition of $\mathcal{R}$ and (10.4) that $U_{\mathrm{HMM}} \in V_H$ is the solution of (10.6) if and only if

$$\hat{a}(U_0; U_0 - U_{\mathrm{HMM}}, W) = \mathcal{R}(U_0, U_{\mathrm{HMM}}, W) + [a(U_0, W) - (f, W)] + [a_H(U_{\mathrm{HMM}}, W) - a(U_{\mathrm{HMM}}, W)] \quad \forall W \in V_H. \tag{10.9}$$

Define

$$E(V, W) = a_H(v, w) - a(V, W) \quad \forall V,\ W \in V_H,$$

and

$$\overline{E} = \max_{V \in U_H \cap W^{1,\infty}(\Omega), W \in V_H} \frac{|E(v, w)|}{|V|_{1,\Omega}|W|_{1,\Omega}}.$$

10.1. Existence and uniqueness of a solution

To prove existence and uniqueness of a solution to (10.6), we introduce the projection of U_0 into V_H through the linearized bilinear form $\hat{a}$:

$$\hat{a}(U_0; P_H U_0, V) = \hat{a}(U_0; U_0, V) \quad \forall V \in V_H. \tag{10.10}$$

It follows from (10.7) that $P_H U_0$ exists and is unique for $0 < H < H_0$, and satisfies [47]

$$\|U_0 - P_H U_0\|_{1,\infty,\Omega} \le CH, \qquad \|U_0 - P_H U_0\|_{1,\Omega} \le CH, \tag{10.11}$$

if $U_0 \in W^{2,\infty}(\Omega)$. When $U_0 \in W^{2,p}(\Omega)$ $(p > d)$, it holds that

$$\|U_0 - P_H U_0\|_{1,\infty,\Omega} \le CH^{1-d/p}. \tag{10.12}$$

Finally, for a given $x \in \Omega$, we define the discrete Green's function $G_H^x \in V_H$ by

$$\hat{a}(U_0; v, G_H^x) = \partial V(x) \quad \text{a.e. in } \Omega,\ V \in V_H, \tag{10.13}$$

where ∂V indicates any of the partial derivatives $\partial V/\partial x_i$ $(i = 1, 2, \ldots, d)$. This function satisfies

$$\|G_H^x\|_{1,1,\Omega} \le C|\ln H|. \tag{10.14}$$

Theorem 10.1. *Assume that L_1 is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ and $U_0 \in U \cap W^{2,p}(\Omega)$ with $p > d$. In addition, assume that $\overline{E}$ is bounded and there are constants C_1 and H_1 such that for $0 < H \le H_1$,*

$$\overline{E}^{1/2}|\ln H| \le C_1. \tag{10.15}$$

Then problem (10.6) has a solution U_{HMM} satisfying

$$\begin{aligned} &\|U_{\mathrm{HMM}} - P_H U_0\|_{1,\infty,\Omega} \le \overline{E}^{1/2} + H^{1-d/p}, \\ &\|U_{\mathrm{HMM}} - U_0\|_{1,\infty,\Omega} \le C\left(\overline{E}^{1/2} + H^{1-d/p}\right). \end{aligned} \tag{10.16}$$

Furthermore, if, for all $V_1, V_2, W \in V_H$ with $\|V_1\|_{1,\infty,\Omega} + \|V_2\|_{1,\infty,\Omega} \le M$, there is a constant $\zeta_0(M)$, with $0 < \zeta_0 < 1$, such that

$$|E(V_1, W) - E(V_2, W)| \le \zeta_0(M)\|V_1 - V_2\|_{1,\Omega}\|W\|_{1,\Omega}, \tag{10.17}$$

then this solution U_{HMM} is locally unique.

Proof. We define the nonlinear mapping $\mathcal{L}_1 : V_H \to V_H$ by

$$\hat{a}(U_0; \mathcal{L}_1(V), W) = \hat{a}(U_0; U_0, W) - \mathcal{R}(U_0, V, W) + a(V, W) - a_H(V, W) \quad \forall W \in V_H.$$

This mapping is continuous using (10.7) and (10.8). We also define the set

$$B = \left\{V \in V_H : \|V - P_H U_0\|_{1,\infty,\Omega} \le \overline{E}^{1/2} + H^{1-d/p}\right\}.$$

Note that, by (10.10),

$$\hat{a}(U_0; \mathcal{L}_1(V) - P_H U_0, W) = -\mathcal{R}(U_0, V, W) + a(V, W) - a_H(V, W) \quad \forall W \in V_H.$$

Choosing $W = G_H^x$ in this equation and applying (10.8), (10.12), (10.14) and (10.15), we see that, with $V \in B$ and $\|V\|_{1,\infty,\Omega} \le M$,

$$\begin{aligned}\|\mathcal{L}_1(v) - P_H U_0\|_{1,\infty,\Omega} &\le C(M)\left(\|U_0 - V\|^2_{1,\infty,\Omega} + \overline{E}\right)|\ln H| \\ &\le C(M)\left(\|U_0 - P_H U_0\|^2_{1,\infty,\Omega} + \|P_H U_0 - v\|^2_{1,\infty,\Omega} + \overline{E}\right)|\ln H| \\ &\le C(M)\left(H^{2-2d/p} + \overline{E}\right)|\ln H|.\end{aligned}$$

Because $V \in B$ and $\overline{E}$ is bounded (e.g., $\overline{E} \le \mathcal{C}_1$), we see that

$$\begin{aligned}\|V\|_{1,\infty,\Omega} &\le \|V - P_H U_0\|_{1,\infty,\Omega} + \|P_H U_0\|_{1,\infty,\Omega} \\ &\le \overline{E}^{1/2} + C(U_0) \le \mathcal{C}_1^{1/2} + C(U_0) \equiv \mathcal{C}_0.\end{aligned} \tag{10.18}$$

Combining these two inequalities, we have

$$\|\mathcal{L}_1(V) - P_H U_0\|_{1,\infty,\Omega} \le C(\mathcal{C}_0)\left(H^{2-2d/p} + \overline{E}\right)|\ln H|.$$

Defining $C_1 = 1/C(\mathcal{C}_0)$, it follows from (10.15) that

$$\|\mathcal{L}_1(V) - P_H U_0\|_{1,\infty,\Omega} \le \overline{E}^{1/2} + C(\mathcal{C}_0)H^{2-2d/p}|\ln H|.$$

Thus there is a constant H_2 such that for $0 < H \le H_2$, we obtain

$$\|\mathcal{L}_1(V) - P_H U_0\|_{1,\infty,\Omega} \le \overline{E}^{1/2} + H^{1-d/p}.$$

Set $H_1 = \min(H_0, H_2)$. Then, for $0 < H \le H_1$, we see that $\mathcal{L}_1(B) \subset B$. The Brouwer fixed point theorem means that there is a $U_{\text{HMM}} \in B$ such that $\mathcal{L}_1(U_{\text{HMM}}) = U_{\text{HMM}}$.

To prove the uniqueness, let U^1_{HMM} and U^2_{HMM} be two solutions of (10.6). Then it follows from (10.7) that, with $U^t_{\text{HMM}} = (1-t)U^2_{\text{HMM}} + tU^1_{\text{HMM}}$,

$$\begin{aligned}C\|U^1_{\text{HMM}} - U^2_{\text{HMM}}\|_{1,\Omega} &\le \sup_{W \in U_H} \frac{\int_0^1 \hat{a}(U^t_{\text{HMM}}; U^1_{\text{HMM}} - U^2_{\text{HMM}}, W)\mathrm{d}t}{\|W\|_{1,\Omega}} \\ &\le \sup_{W \in V_H} \frac{|a(U^1_{\text{HMM}}, W) - a(U^2_{\text{HMM}}, W)|}{\|W\|_{1,\Omega}}.\end{aligned}$$

Note that, by (10.6),

$$a(U^1_{\text{HMM}}, W) - a(U^2_{\text{HMM}}, W) = \left[a(U^1_{\text{HMM}}, W) - a_H(U^1_{\text{HMM}}, W)\right] - \left[a(U^2_{\text{HMM}}, W) - a_H(U^2_{\text{HMM}}, W)\right].$$

Because both U^1_{HMM} and U^2_{HMM} are in the set B, it follows from (10.18) that $\|U^1_{\text{HMM}}\|_{1,\infty,\Omega} + \|U^2_{\text{HMM}}\|_{1,\infty,\Omega} \le 2\mathcal{C}_0$, which, together with (10.17), implies

$$\|U^1_{\text{HMM}} - U^2_{\text{HMM}}\|_{1,\Omega} \le \zeta_0(2\mathcal{C}_0)\|U^1_{\text{HMM}} - U^2_{\text{HMM}}\|_{1,\Omega}.$$

Since $\zeta_0 < 1$ via assumption, $U^1_{\text{HMM}} = U^2_{\text{HMM}}$. Therefore, the solution U_{HMM} is locally unique. □

10.2. Error estimates

The multiscale finite element solution U_{HMM} in the next theorem refers to the one that satisfies the conditions in Theorem 10.1.

Theorem 10.2. *Let U_0 and U_{HMM} be the solutions of (10.4) and (10.6), respectively, and $U_0 \in W^{2,\infty}(\Omega)$. Then there is $H_0 > 0$ such that for $0 < H < H_0$,*

$$\|U_0 - U_{\text{HMM}}\|_{1,\Omega} \le C(H + \epsilon), \qquad \|U_0 - U_{\text{HMM}}\|_{1,\infty,\Omega} \le C(H + \epsilon)|\ln H|, \tag{10.19}$$

provided that $\epsilon|\ln H|$ is sufficiently small.

Proof. Taking $W = P_H U_0 - U_{\text{HMM}}$ in (10.9) and using (10.7) and (10.8), and a similar argument as in fourth section, we see that

$$\|P_H U_0 - U_{\text{HMM}}\|_{1,\Omega} \le C\left(\|U_0 - U_{\text{HMM}}\|^2_{1,4,\Omega} + \overline{E} + H\right). \tag{10.20}$$

As in the conforming case [4], it can be shown that

$$\overline{E} \le C\epsilon, \tag{10.21}$$

provided that $\epsilon|\ln H|$ is sufficiently small. Also, applying an interpolation inequality, we have

$$\|U_0 - U_{\text{HMM}}\|_{1,4,\Omega}^2 \le \|U_0 - U_{\text{HMM}}\|_{1,\Omega} \|U_0 - U_{\text{HMM}}\|_{1,\infty,\Omega}. \tag{10.22}$$

Consequently, combining (10.20)–(10.22) and using (10.11) and Theorem 10.1 yields the first result in (10.19).

Choosing $W = G_H^x$ in (10.9) and using (10.8), (10.11) and (10.14), we see that

$$\|P_H U_0 - U_{\text{HMM}}\|_{1,\infty,\Omega} \le C\overline{E}|\ln H|\|P_H U_0 - U_{\text{HMM}}\|_{1,\infty,\Omega} + C\left(\|U_0 - U_{\text{HMM}}\|_{1,\infty,\Omega}^2 + \overline{E} + H\right)|\ln H|.$$

Applying (10.21), if $\epsilon \ln H|$ is sufficiently small, it follows that

$$\|P_H U_0 - U_{\text{HMM}}\|_{1,\infty,\Omega} \le C\left(\|U_0 - U_{\text{HMM}}\|_{1,\infty,\Omega}^2 + \overline{E} + H\right)|\ln H|.$$

As a result, applying Theorem 10.1, we obtain

$$\|U_0 - U_{\text{HMM}}\|_{1,\infty,\Omega} \le C\left(\|P_H U_0 - U_0\|_{1,\infty,\Omega} + \left(\overline{E} + H\right)|\ln H|\right),$$

which, together with (10.11), implies the second inequality in (10.19). □

11. A random homogenization problem

In the previous sections we have assumed that the coefficient a_ϵ in problem (2.1) has the form $a(x, x/\epsilon)$ and $a(x, y)$ is periodic in y. In many problems such as in porous media flows [48], this coefficient is often random. In this section we indicate how to extend the multiscale finite element analysis performed for (2.1) to a multiscale problem with a random coefficient. Again, as example, we consider the nonconforming finite element HMM, following [4] for the conforming finite element methods.

Let (D, F, P) be a probability space and $a(y, \omega) = \left(a_{ij}(y, \omega)\right)$ be a random field, $y \in \mathbb{R}^d, \omega \in D$, whose statistics is invariant under integer shifts. Furthermore, let a satisfy the uniform ellipticity condition (2.2); i.e.,

$$a_*|\zeta|^2 \le \sum_{i,j=1}^d a_{ij}(y, \omega)\zeta_i\zeta_j \le a^*|\zeta|^2 \quad \forall \omega \in D,\ y,\ \zeta \in \mathbb{R}^d, \tag{11.1}$$

for some positive constants a_* and a^*. Problem (2.1) now takes the form

$$\begin{aligned} &-\nabla \cdot \left(a(x/\epsilon, \omega)\nabla u^\epsilon\right) = f \quad \text{in } \Omega, \\ &u^\epsilon = 0 \quad \text{on } \Gamma. \end{aligned} \tag{11.2}$$

As in (3.10), let χ^j satisfy [49]

$$-\nabla_y \cdot \left(a(y, \omega)\nabla_y \chi^j\right) = \sum_{i=1}^d \frac{\partial}{\partial y_i} a_{ij}(y, \omega), \tag{11.3}$$

and $\nabla\chi^j$ is assumed to be stationary under integer shifts. χ^j is generally not stationary. Define the average operator with respect the measure P (mathematical expectation)

$$\langle v\rangle = \mathbb{E}\int_{[-1/2,1/2]^d} v(y)\mathrm{d}y.$$

The homogenized coefficient A is given by

$$A = \langle a(I + \nabla\chi)\rangle, \tag{11.4}$$

where I is the identity matrix and $\chi = (\chi^1, \chi^2, \ldots, \chi^d)^{\mathrm{T}}$. With this coefficient, the variational formulation of the homogenized problem is defined as in (3.8).

For the convergence analysis in the random case, we will use an important mixing condition [50]. For a subdomain $B \subset \mathbb{R}^d$, denote by $\Phi(B)$ the σ-algebra generated by the parameters $\{a(y, \omega) : y \in B\}$. Let ζ_1 and ζ_2 be two random variables that are measurable with respect to $\Phi(B_1)$ and $\Phi(B_2)$, respectively. We assume that

$$|\mathbb{E}(\zeta_1\zeta_2) - \mathbb{E}(\zeta_1)\mathbb{E}(\zeta_2)| \le \mathrm{e}^{-C\mathrm{dist}(B_1,B_2)}\sqrt{\mathbb{E}\zeta_1^2}\sqrt{\mathbb{E}\zeta_2^2}. \tag{11.5}$$

This type of exponential decay condition is often used for geostatistical models.

For the barycenter x_T of each $T \in T_H$, Δ_T introduced in the fourth section is replaced with

$$\Delta_T(\delta) = x_T + \delta Y,$$

where the constant δ should be a few times greater than the local correlation length in a_ϵ. Below we write $\Delta_T = \Delta_T(\delta)$ in short. Let $V_H \subset L^2(\Omega)$ be the nonconforming finite element space defined in the fourth section. The local problem (4.1) becomes: For any $V \in V_H$, $v_T^\epsilon \in H^1(\Delta_T)$ satisfies

$$\begin{aligned} &\left(a(x_T, x/\epsilon)\nabla v_T^\epsilon, \nabla w\right)_{\Delta_T} = 0 \quad \forall w \in H_0^1(\Delta_T), \\ &v_T^\epsilon = V \quad \text{on } \partial\Delta_T. \end{aligned} \tag{11.6}$$

Then, for any $U,\ V \in V_H$, we introduce the bilinear form

$$a_H(U, V) = \sum_{T\in T_H} \frac{|T|}{(\delta/2)^d} \left(a_\epsilon \nabla u_T^\epsilon, \nabla v_T^\epsilon\right)_{\Delta_T},$$

where u_T^ϵ corresponds to $U \in V_H$ through (11.6). The reason to use $\delta/2$, instead of δ in the bilinear form $a_H(\cdot,\cdot)$, is to reduce the effect of the imposed boundary condition on $\partial\Delta_T$ [4]. Now, the nonconforming finite element HMM for (11.2) is to seek $U_{\text{HMM}} \in V_H$ such that

$$a_H(U_{\text{HMM}}, V) = (f, V) \quad \forall V \in V_H. \tag{11.7}$$

We now state error estimates in the random case for which we also need Strang's second lemma:

Lemma 11.1. *Let U_{HMM} and U_0 be the respective solutions of (3.8) and (11.7), with A given by (11.4). Then there is a constant $C > 0$, independent of H and ϵ, such that*

$$|\Pi_H U_0 - U_{\text{HMM}}|_H \le C\left\{\inf_{V\in V_H} |\Pi_H U_0 - V|_H + \sup_{W\in V_H, W\neq 0} \frac{|a_H(\Pi_H U_0, W) - (f, W)|}{|W|_H}\right\}, \tag{11.8}$$

where Π_H is the standard interpolation operator into V_H.

We remark that the homogenization results in [51] will be used below and may be overestimated because they are based on the Green function estimates that are not required for the computation of effective coefficients. Because of this, the next convergence result here may be overestimated as well.

Theorem 11.2. *Let U_{HMM} and U_0 be the respective solutions of (3.8) and (11.7), where the homogenized coefficient A is now given by (11.4), and $U_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$. Then, under condition (11.5), we have*

$$\begin{aligned} &\mathbb{E}\,|U_0 - U_{\text{HMM}}|_H \le C(\kappa)\left(H + \left(\frac{\epsilon}{\delta}\right)^\kappa\right), \\ &\mathbb{E}\,\|U_0 - U_{\text{HMM}}\|_{0,\Omega} \le C(\kappa)\left(H^2 + \left(\frac{\epsilon}{\delta}\right)^\kappa\right), \end{aligned} \tag{11.9}$$

where

$$\kappa = \begin{cases} \dfrac{6-12\lambda}{25-8\lambda} & \text{if } d = 3, \\ \dfrac{1}{2} & \text{if } d = 1, \end{cases}$$

for any $0 < \lambda < 1/2$.

Proof. Again, because the conforming P_1 finite element space is a subspace of V_H, it follows [4,51] that

$$\mathbb{E}\inf_{V\in V_H} |\Pi_H U_0 - V|_H \le \mathbb{E}|\Pi_H U_0 - U_0|_H + \mathbb{E}\inf_{V\in V_H} |U_0 - V|_H \le C(\kappa)\left(H + \left(\frac{\epsilon}{\delta}\right)^\kappa\right). \tag{11.10}$$

Also, the consistency error in (11.8) can be estimated by combining the techniques in the fourth section and those for handling the conforming finite element HMM for a random homogenization problem (see Appendix A in [4]). □

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