Adv Comput Math (2013) 38:281–320
DOI 10.1007/s10444-011-9237-9

On the semi-discrete stabilized finite volume method for the transient Navier–Stokes equations

Jian Li · Zhangxin Chen

Received: 2 July 2011 / Accepted: 13 September 2011 /
Published online: 11 October 2011

Abstract A stabilized finite volume method for solving the transient Navier–Stokes equations is developed and studied in this paper. This method maintains conservation property associated with the Navier–Stokes equations. An error analysis based on the variational formulation of the corresponding finite volume method is first introduced to obtain optimal error estimates for velocity and pressure. This error analysis shows that the present stabilized finite volume method provides an approximate solution with the same convergence rate as

Communicated by Zhongying Chen.

This research was supported in part by the National Science Foundation of China (No. 11071193), Natural Science New Star of Science and Technologies Research Plan in Shaanxi Province of China (No. 2011kjxx12), Research Program of Education Department of Shaanxi Province (No. 11JK0490), the project-sponsored by SRF for ROCS, SEM, and NSERC/AERI/Foundation CMG Chair and iCORE Chair Funds in Reservoir Simulation.

J. Li (✉)
Department of Mathematics, Baoji University of Arts and Sciences,
Baoji 721013, People's Republic of China
e-mail: jiaaanli@gmail.com

J. Li · Z. Chen
Department of Chemical & Petroleum Engineering, Schulich School of Engineering,
University of Calgary, 2500 University Drive N.W., Calgary,
AB T2N 1N4, Canada

Z. Chen
e-mail: zhachen@ucalgary.ca

Z. Chen
Faculty of Science, Xi'an Jiaotong University, Xi'an, 710049,
People's Republic of China

that provided by the stabilized linear finite element method for the Navier–Stokes equations under the same regularity assumption on the exact solution and a slightly additional regularity on the source term. The stability and convergence results of the proposed method are also demonstrated by the numerical experiments presented.

Keywords Navier–Stokes equations · Stabilized finite volume method · *inf-sup* condition · Local pressure projection · Optimal error estimate

Mathematics Subject Classifications (2010) 35Q10 · 65N30 · 76D05

1 Introduction

Finite difference, finite element, and finite volume methods are three major numerical methods for solving partial differential equations. Among them, the finite volume method is the most intuitive one because it is based on local mass, momentum, or energy conservation over volumes (control volumes or co-volumes) in practical applications. It lies somewhere between the finite element and finite difference methods; it has a flexibility similar to that of the finite element method in handling complicated geometries, and its implementation is comparable to that of the finite difference method.

The finite volume method is also known as the control volume method, the covolume method, or the first-order generalized difference method. Much work was devoted to its error analysis for second-order elliptic and parabolic partial differential equations [1, 4–9, 14, 15, 30, 32, 33, 38, 39]. The H^1 error estimate was first given on the triangle grids for second order elliptic partial differential equations [31]. Error estimates of optimal order in the H^1-norm are the same as those for the linear finite element method. Moreover, error estimates of optimal order in the L^2-norm can also be obtained. The finite volume method for the Stokes equations was studied as well [11, 25, 37, 40]. It was analyzed through a relationship to the finite element method, and its error estimates were obtained through those known for the latter. However, for the Stokes equations only the finite element pairs that satisfy a discrete *inf-sup* condition for velocity and pressure were studied.

A stabilized finite element method based on a local pressure projection for the Stokes equations has recently been developed [2, 3, 13, 26]. This method is free of stabilization parameter, does not require any calculation of high-order derivatives or edge-based data structure, and can be implemented at the element level [2, 3, 13]. It is known that this element pair does not satisfy the discrete *inf-sup* condition for the Stokes equations. Nevertheless, it is of practical importance in real applications. In particular, it is efficient and simple in terms of implementation. In addition to the above features, another important feature is that this stabilized method also has superconvergent results [20, 26, 27], which is in strong contrast with other stabilized methods.

In this paper we develop and study a stabilized finite volume method for the transient Navier–Stokes equations. The finite volume method presented here is instead designed to inherit a local conservation property associated with the differential equations. There exist two major difficulties in the convergence analysis of this stabilized finite volume method. While the analysis can be carried out through its relationship with the conforming finite elements of the lowest-equal order pair as for the Stokes equations [25], there exists additional difficulty in the treatment of the nonlinear term (i.e., trilinear term) appearing in the finite volume formulation of the Navier–Stokes equations. In particular, the trilinear form introduced in the finite volume method does not have the same anti-symmetric property as that in the finite element method [19, 35]. The other major difficulty is associated with the analysis of the discretization of the transient term. Because an equivalent operator between the finite volume method and the $P_1 - P_1$ pair of the conforming finite elements is used, this operator necessarily appears in the transient term, and the resulting discretization cannot be treated by using the standard parabolic argument [10, 36]. To overcome these difficulties, a finite volume projection based on the variational formulation of the corresponding finite volume method is first introduced for the Stokes equations. A key argument in the present analysis is to combine this projection and a finite element projection for the Stokes equations without any additional regularity on the exact solution. Furthermore, some results related to the equivalence between the standard L^2-norm and the norm induced by the above mentioned equivalent operator [7] will be used. These techniques, together with the introduction of a duality argument for the derivation of the L^2-error estimate for velocity, will yield convergence rates of optimal order for the present stabilized finite volume method. Compared with the results in [22], the main contribution of this paper is to establish optimal estimates for the Navier–Stokes equations based on local conservation property. Recently, there are lots of wonderful jobs by Eymard et al. [16–18] on the mathematical properties and convergence analysis of the collocated clustered finite volume scheme for the incompressible flows. However, it still requires much research on theoretical analysis of the finite volume method for the transient Navier–Stokes equations.

The rest of the paper is organized as follows: In the next section, we introduce some notation and the transient Navier–Stokes equations. Then, in Section 3, a stabilized finite element method for these equations is recalled. The stabilized finite volume method is defined in Section 4, and some useful lemmas are shown. Stability and optimal order estimates for this method are obtained in Sections 5 and 6. Finally, numerical results to check the theoretical results obtained are provided in Section 7.

2 Function settings

Let Ω be a bounded domain in $\Re^2$, with a Lipschitz-continuous boundary Γ, satisfying a further condition as stated in (A1) below. The transient Navier–Stokes equations are

$$u_t - \nu\Delta u + \nabla p + (u\cdot\nabla)u + \frac{1}{2}(\operatorname{div} u)u = f,\ \operatorname{div} u = 0, \quad (x,t)\in\Omega\times(0,T], \tag{2.1}$$

$$u(x,0) = u_0(x), \quad x\in\Omega, \quad u(x,t)|_\Gamma = 0, \quad t\in[0,T], \tag{2.2}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t))$ represents the velocity vector, $p = p(x,t)$ the pressure, $f = f(x,t)$ the prescribed body force, $\nu > 0$ the viscosity, $T > 0$ the final time, and $u_t = \partial u/\partial t$. Note that the term $(\operatorname{div} u)u/2$ is added to ensure the dissipativity of the Navier–Stokes equations [35]. To introduce a variational formulation of (2.1) and (2.2), set

$$X = (H_0^1(\Omega))^2, \quad Y = (L^2(\Omega))^2, \quad M = L_0^2(\Omega) = \left\{q\in L^2(\Omega);\ \int_\Omega q\,dx = 0\right\},$$

$$V = \{v\in X : \operatorname{div} v = 0\}, \quad D(A) = (H^2(\Omega))^2\cap V,$$

where A indicates the Laplace operator.

As noted, a further assumption on the domain Ω is needed:

(A1) Assume that Ω is regular in the sense that the unique solution $(v,q)\in(X,M)$ of the stationary Stokes equations

$$-\Delta v + \nabla q = g, \quad \operatorname{div} v = 0 \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0$$

for a prescribed $g\in Y$ exists and satisfies

$$\|v\|_2 + \|q\|_1 \le c\|g\|_0,$$

where $c > 0$ is a constant depending only on Ω and $\|\cdot\|_i$ denotes the usual norm of the Sobolev space $H^i(\Omega)$ or $(H^i(\Omega))^2$ for $i = 0,1,2$. Below the positive constants c and $c_i,\ i = 0,1,2\cdots$, will depend at most on the data (ν, T, u_0, Ω, f).

We denote by $(\cdot,\cdot)$ and $\|\cdot\|_0$ the inner product and norm on $L^2(\Omega)$ or $(L^2(\Omega))^2$, as appropriate. The spaces $H_0^1(\Omega)$ and X are equipped with their usual scalar product and norm

$$((u,v)) = (\nabla u, \nabla v), \quad \|u\|_1 = ((u,u))^{1/2}.$$

(Due to the norm equivalence between $\|u\|_1$ and $\|\nabla u\|_0$ on $H_0^1(\Omega)$, we are using the same notation for them.) It is well known that for each $v\in X$ there hold the following inequalities:

$$\|v\|_{L^4} \le 2^{1/4}\|v\|_0^{1/2}\|v\|_1^{1/2}, \quad \|v\|_0 \le c_1\|v\|_1. \tag{2.3}$$

The next assumption will be also used in the error analysis.

(A2) The initial velocity $u_0 \in D(A)$ and the body force $f(x,t) \in L^2(0,T;(H^1(\Omega))^2)$ are assumed to satisfy

$$\|u_0\|_2 + \left(\int_0^T \left(\|f\|_1^2 + \|f_t\|_0^2\right)\,dt\right)^{1/2} \leq c.$$

The continuous bilinear forms $a(\cdot,\cdot)$ on $X \times X$ and $d(\cdot,\cdot)$ on $X \times M$ are, respectively, defined by

$$a(u,v) = \nu((u,v)) \qquad \forall u, v \in X, \quad d(v,q) = -(v, \nabla q) = (q, \operatorname{div} v) \quad \forall v \in X,\ q \in M,$$

and the generalized bilinear form $\mathcal{B}(\cdot;\cdot)$ on $(X,M)\times(X,M)$ is given by

$$\mathcal{B}((u,p);(v,q)) = a(u,v) - d(v,p) + d(u,q), \quad (u,p),\ (v,q) \in (X,M).$$

The latter form satisfies the following inequalities [35]:

$$|\mathcal{B}((u,p);(u,p)) = \nu\|u\|_1^2, \tag{2.4}$$

$$|\mathcal{B}((u,p);(v,q))| \leq c(\|u\|_1 + \|p\|_0)(\|v\|_1 + \|q\|_0), \tag{2.5}$$

$$\beta(\|u\|_1 + \|p\|_0) \leq \sup_{(v,q)\in(X,M)} \frac{|\mathcal{B}((u,p);(v,q))|}{\|v\|_1 + \|q\|_0}, \tag{2.6}$$

for all $(u,p),\ (v,q) \in (X,M)$, where β is a positive constant. Also, the trilinear term $b(\cdot,\cdot,\cdot)$ on $X \times X \times X$ is defined by [35]

$$b(u,v,w) = ((u\cdot\nabla)v, w) + \frac{1}{2}((\operatorname{div} u)v, w) \qquad \forall u, v, w \in X.$$

It satisfies [21, 35]

$$b(u,v,v) = 0, \tag{2.7}$$

$$|b(u,v,w)| + |b(w,v,u)| + |b(u,w,v)| \leq c_2\|u\|_0^{1/2}\|u\|_1^{1/2}\left(\|v\|_1\|w\|_0^{1/2}\|w\|_1^{1/2} + \|v\|_0^{1/2}\|v\|_1^{1/2}\|w\|_1\right), \tag{2.8}$$

for all $u, v, w \in X$, and

$$|b(u,v,w)| + |b(v,u,w)| + |b(w,u,v)| \leq c_2\|u\|_1\|v\|_2\|w\|_0, \tag{2.9}$$

for all $u \in X$, $v \in D(A)$, $w \in Y$.

Here, the idea is used to easily achieve optimal results in analyzing the finite element discretization by adding a useful term $\frac{1}{2}((\operatorname{div} u)v, w)$ in the trilinear term. The trilinear term defined above is still consistent in original problem.

The mixed variational form of (2.1) and (2.2) is to seek $(u,p) \in (X,M)$, $t > 0$, such that, for all $(v,q) \in (X,M)$,

$$(u_t, v) + \mathcal{B}((u,p);(v,q)) + b(u,u,v) = (f,v), \tag{2.10}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{2.11}$$

For the subsequent convenience, we recall the Gronwall Lemma that will be frequently used.

Lemma 2.1 [34] *Let $g(t)$, $\ell(t)$, and $\xi(t)$ be three nonnegative functions satisfying, for $t \in [0, T]$,*

$$\xi(t) + G(t) \leq c + \int_0^t \ell \, ds + \int_0^t g\xi \, ds,$$

where $G(t)$ is a nonnegative function on $[0, T]$. Then

$$\xi(t) + G(t) \leq \left(c + \int_0^t \ell \, ds \right) \exp \left(\int_0^t g \, ds \right). \tag{2.12}$$

The following result is concerned with the existence, uniqueness, and regularity of a global strong solution to the Navier–Stokes equations (2.1) and (2.2).

Lemma 2.2 [22] *Assume that (A1) and (A2) hold. Then, for any given $T > 0$ there exists a unique solution (u, p) of (2.1) and (2.2) satisfying the following regularities:*

$$\sup_{0<t\leq T} \left(\|u(t)\|_2^2 + \|p(t)\|_1^2 + \|u_t(t)\|_0^2 \right) + \int_0^T \left(\|u_t(s)\|_1^2 + \|p_t(s)\|_0^2 \right) ds \leq c, \tag{2.13}$$

$$\sup_{0<t\leq T} \tau(t)\|u_t(t)\|_1^2 + \int_0^T \tau(s) \left(\|u_t(s)\|_2^2 + \|p_t(s)\|_1^2 + \|u_{tt}(s)\|_0^2 \right) ds \leq c, \tag{2.14}$$

where $\tau(t) = \min\{1, \ t\}$.

3 A stabilized finite element method

For $h > 0$, let K_h be a triangulation of Ω into triangles, assumed to be shape-regular in the usual sense [10, 12, 19]. Associated with K_h we introduce finite dimensional subspaces $(X_h, M_h) \subset (X, M)$. For these spaces we assume that the following approximation properties hold: For $(v, q) \in (D(A), H^1(\Omega) \cap M)$, there exist approximations $I_h v \in X_h$ and $J_h q \in M_h$ such that

$$\|v - I_h v\|_0 + h\|v - I_h v\|_1 \leq ch^2\|v\|_2, \tag{3.1}$$

$$\|q - J_h q\|_0 + h\|q - J_h q\|_1 \leq ch\|q\|_1, \tag{3.2}$$

where the L^2-projection $J_h : M \to M_h$ satisfies

$$(p - J_h p, q_h) = 0 \qquad \forall \ p \in M, \ q_h \in M_h.$$

Also, we assume that the inverse inequality holds [10, 12]

$$\|v_h\|_1 \le c_3 h^{-1}\|v_h\|_0, \quad \|v_h\|_\infty \le c_4|\log h|^{1/2}\|v_h\|_1 \quad \forall v_h \in X_h. \tag{3.3}$$

This paper will focus on the analysis of the lowest equal-order pair of finite elements for velocity and pressure:

$$X_h = \left\{v_h \in \left(C^0(\Omega)\right)^2 \cap X : \; v_h|_K \in \left(P_1(K)\right)^2 \quad \forall K \in K_h\right\},$$

$$M_h = \{q_h \in C^0(\Omega) \cap M : \; q_h|_K \in P_1(K) \quad \forall K \in K_h\},$$

where $P_1(K)$ is the set of linear functions on element K.

Many stable pairs of finite element spaces (X_h, M_h) have been proposed in the existing literature. Examples include Taylor–Hood element $P_2 - P_1$ and MINI-element $P_1b - P_1$, etc. Especially, Taylor–Hood element performs a superconvergence result since it employs the higher order finite element pair than others. Also, MINI-element pair is a implicit stabilized finite element method on uniformly mesh for the incompressible flows. In fact, there are also many quadrilateral finite element pairs are to be preferred for the incompressible flows. However, it is still open problem for optimal theoretic analysis of finite volume method for the stationary Stokes equations.

In this paper, the stabilized $P_1 - P_1$ element has several important features: it is more accurate and convenient to approximate both the velocity and pressure with the same number of degrees of freedom. Furthermore, it has a very simple data structure due to the use of the same type of nodal values for velocities and pressure which allows for an efficient vectorization of solution processes. Moreover, standard multigrid techniques can be used for solving the algebraic systems with good efficiency. To stabilized the lowest equal-order finite element pair, a stabilized fintie element method is applied by local difference between a consistent matrix and an underintegrated mass matrix to stabilize it [2, 3, 13, 26, 28]:

$$G(p_h, q_h) = \bar{p}(M_k - M_1)\bar{q}^T = \bar{p}M_k\bar{q}^T - \bar{p}M_1\bar{q}^T,$$

where

$$\bar{p} = [p_0, p_1, \ldots, p_{N-1}], \quad \bar{q}^T = [q_0, q_1, \ldots, q_{N-1}]^T,$$
$$M_{ij} = (\phi_i, \phi_j), \quad p_h = \sum_{i=0}^{N-1} p_i\phi_i,$$
$$p_i = p_h(x_i)\forall p_h \in M_h, \; i, j = 0, 1, \ldots, N-1,$$

ϕ_i is a basis function of the pressure space M_h such that its value is unity at node x_i and zero at other nodes, and the symmetric positive definite pressure mass matrices M_k ($k \ge 2$) and M_1 are computed by using the kth-order and first order Gauss integrals in each spatial direction, respectively. In addition, p_i and q_i, $i = 0, 1 \ldots, N-1$, are the values of p_h and q_h at the node x_i, and $\bar{q}^T$ is the transpose of the vector $\bar{q}$.

Let $\Pi_h: L^2 \rightarrow R_h$ be the standard L^2-projection with the following properties:

$$(p, q_h) = (\Pi_h p, q_h) \qquad \forall p \in M,\ q_h \in R_h, \tag{3.4}$$

$$\|\Pi_h p\|_0 \leq c_5 \|p\|_0 \qquad \forall p \in M, \tag{3.5}$$

$$\|p - \Pi_h p\|_0 \leq c_6 h \|p\|_1 \qquad \forall p \in H^1(\Omega) \cap M, \tag{3.6}$$

where $R_h = \{q_h \in M : q_h|_K$ is a constant for each $K \in K_h\}$. Then we can formulate the bilinear form $G(\cdot, \cdot)$ as

$$G(p, q) = (p - \Pi_h p, q - \Pi_h q). \tag{3.7}$$

The L^2-projection operator Π_h can similarly be defined in the vector case. The bilinear form $G(\cdot, \cdot)$ in (3.7) is a symmetric positive semi-definite form generated on each local element K.

Using the above notation, the stabilized finite element formulation of system (2.10) and (2.11) reads: Find $(\bar{u}_h, \bar{p}_h) \in (X_h, M_h)$, $t \in [0, T]$, such that, for all $(v_h, q_h) \in (X_h, M_h)$,

$$(\bar{u}_{ht}, v_h) + \mathcal{B}_h((\bar{u}_h, \bar{p}_h); (v_h, q_h)) + b(\bar{u}_h, \bar{u}_h, v_h) = (f, v_h), \tag{3.8}$$

$$u_h(0) = u_{0h}, \tag{3.9}$$

where u_{0h} is some approximation of u_0 in X_h satisfying the approximation property (3.1) and

$$\mathcal{B}_h((\bar{u}_h, \bar{p}_h); (v_h, q_h)) = a(\bar{u}_h, v_h) - d(v_h, \bar{p}_h) + d(\bar{u}_h, q_h) + G(\bar{p}_h, q_h)$$

is the stabilized bilinear form. The following theorem establishes the continuity and weak coercivity of (3.8) for the equal-order finite element pair $P_1 - P_1$ [2, 3, 26]:

Theorem 3.1 *Let (X_h, M_h) be defined as above. Then there exists a positive constant β, independent of h, such that*

$$|\mathcal{B}_h((u, p); (v, q))| \leq c(\|u\|_1 + \|p\|_0)(\|v\|_1 + \|q\|_0) \qquad \forall (u, p),\ (v, q) \in (X, M), \tag{3.10}$$

$$\beta(\|u_h\|_1 + \|p_h\|_0) \leq \sup_{(v_h, q_h) \in (X_h, M_h)} \frac{|\mathcal{B}_h((u_h, p_h); (v_h, q_h))|}{\|v_h\|_1 + \|q_h\|_0} \qquad \forall (u_h, p_h) \in (X_h, M_h), \tag{3.11}$$

$$|G(p, q)| \leq \|p - \Pi_h p\|_0 \|q - \Pi_h q\|_0 \qquad \forall p, q \in M. \tag{3.12}$$

To derive error estimates, we define the Stokes projection operators $(\bar{R}_h, \bar{Q}_h): (X, M) \rightarrow (X_h, M_h)$ by

$$\mathcal{B}_h((\bar{R}_h(v,q), \bar{Q}_h(v,q)); (v_h, q_h)) = B((v,q); (v_h, q_h)) \quad \forall (v,q) \in (X, M),$$

$$(v_h, q_h) \in (X_h, M_h), \tag{3.13}$$

which are well defined and satisfy the following approximation properties:

Lemma 3.2 *Under the assumption of* (A1)*, the projection operators* $(\bar{R}_h, \bar{Q}_h)$ *of the finite element method satisfy*

$$\|v - \bar{R}_h(v,q)\|_0 + h\left(\|v - \bar{R}_h(v,q)\|_1 + \|q - \bar{Q}_h(v,q)\|_0\right) \le ch^2(\|v\|_2 + \|q\|_1), \tag{3.14}$$

for all $(v, q) \in (D(A), H^1(\Omega) \cap M)$.

Proof The proof of Lemma 3.2 is classical and can be easily derived from the classical Galerkin finite element method. More details can be found in [21, 27]. □

The next optimal error estimate holds for the stabilized finite element method (3.8) and (3.9) for the transient Navier–Stokes equations.

Theorem 3.2 [27] *Under the assumptions of* (A1) *and*

$$\|u_0\|_2^2 + \left(\int_0^t \left(\|f\|_0^2 + \|f_t\|_0^2\right) ds\right)^{1/2} \le c, \quad t \in [0, T],$$

it holds that

$$\tau^{1/2}(t)\|u(t) - \bar{u}_h(t)\|_0 + h(\|u(t) - \bar{u}_h(t)\|_1 + \tau^{1/2}(t)\|p(t) - \bar{p}_h(t)\|_0) \le ch^2,$$

$$t \in [0, T]. \tag{3.15}$$

4 A stabilized finite volume method

Let N_h be the set containing all the interior nodes associated with the triangulation K_h, and N be the total number of the nodes. To define the finite volume method, a dual mesh $\tilde{K}_h$ is introduced based on K_h; the elements in $\tilde{K}_h$ are called control volumes. The dual mesh can be constructed by the following rule: For each element $K \in K_h$ with vertices P_j, $j = 1, 2, 3$, select its barycenter Q_j and the midpoint M_j on each of the edges of K, and construct the control volumes in $\tilde{K}_h$ by connecting Q_j to M_j as shown in Fig. 1.

Associated with $\tilde{K}_h$, the dual finite element space is defined as

$$\tilde{X}_h = \left\{\tilde{v} \in \left(L^2(\Omega)\right)^2 : \tilde{v}|_{\tilde{K}} \in P_0(\tilde{K}) \ \forall \tilde{K} \in \tilde{K}_h; \ \tilde{v}|_{\partial \tilde{K}} = 0\right\}.$$

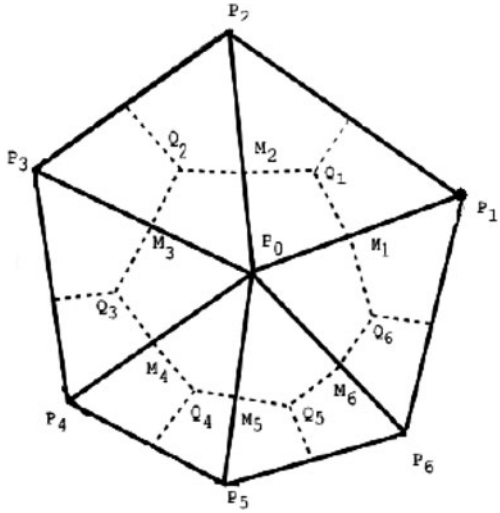


Fig. 1 Control volumes associated with triangles

Obviously, the dimensions of X_h and $\tilde{X}_h$ are the same. Furthermore, there exists an invertible linear mapping $\Gamma_h : X_h \to \tilde{X}_h$ such that for

$$v_h(x) = \sum_{j=1}^{N} v_h(P_j)\varphi_j(x), \qquad x \in \Omega,\ v_h \in X_h, \tag{4.1}$$

we have

$$\Gamma_h v_h(x) = \sum_{j=1}^{N} v_h(P_j)\chi_j(x), \qquad x \in \Omega,\ v_h \in X_h, \tag{4.2}$$

where $\{\varphi_j\}$ indicates the basis of the finite element space X_h and $\{\chi_j\}$ denotes the basis of the finite volume space $\tilde{X}_h$ that are the characteristic functions associated with the dual partition $\tilde{K}_h$:

$$\chi_j(x) = \begin{cases} 1 & \text{if } x \in \tilde{K}_j \in \tilde{K}_h, \\ 0 & \text{otherwise.} \end{cases}$$

The above idea of connecting the trial and test spaces in the Petrov–Galerkin method through the mapping Γ_h was first introduced in [31] in the context of elliptic problems.

To introduce a variational formulation of the finite volume method, we multiply (2.1) by $\Gamma_h v_h \in \tilde{X}_h$ and integrate over the dual elements $\tilde{K} \in \tilde{K}_h$, multiply equation (2.2) by $q_h \in M_h$ and integrate over the primal elements $K \in K_h$, and apply Green's formula to obtain the following bilinear forms:

$$A(u_h, \Gamma_h v_h) = -\sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial \tilde{K}_j} \frac{\partial u_h}{\partial \vec{n}} ds, \quad u_h,\ v_h \in X_h,$$

$$D(\Gamma_h v_h, p_h) = -\sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial \tilde{K}_j} p_h \vec{n}\, ds, \quad p_h \in M_h,$$

$$(f, \Gamma_h v_h) = \sum_{j=1}^{N} v_h(P_j) \cdot \int_{\tilde{K}_j} f\, dx, \quad v_h \in X_h,$$

where $\vec{n}$ is the unit normal outward to $\partial \tilde{K}_j$. Also, we define the trilinear form $b(\cdot, \cdot, \cdot) : X_h \times X_h \times \tilde{X}_h \to \Re$ for the finite volume method

$$b(u_h, v_h, \Gamma_h w_h) = ((u_h \cdot \nabla)v_h, \Gamma_h w_h) + \frac{1}{2}((\mathrm{div} u_h)v_h, \Gamma_h w_h) \quad \forall u_h, v_h, w_h \in X_h. \tag{4.3}$$

Here, the trilinear terms defined in (4.3) still holds the same form as the finite volume method in [29]. However, it does not satisfy anti-symmetric property as (2.7) any more. Thus, it is a key difficulty in theoretic analysis of finite volume method for the nonlinear Navier–Stokes equations.

Now, the stabilized finite volume method is defined for the solution $(u_h, p_h) \in (X_h, M_h)$, $t \in [0, T]$ as follows: For all $(v_h, q_h) \in (X_h, M_h)$,

$$(u_{th}, \Gamma_h v_h) + \mathcal{C}_h((u_h, p_h); (v_h, q_h)) + b(u_h, u_h, \Gamma_h v_h) = (f, \Gamma_h v_h), \tag{4.4}$$

$$u_h(x, 0) = P_h u_0(x), \tag{4.5}$$

where the approximation $u_h \in X_h$ of the initial value u_0 is given as follows:

$$(P_h u_0 - u_0, \Gamma_h v_h) = 0,$$

which satisfies (3.1), and the bilinear form $\mathcal{C}_h(\cdot; \cdot)$ on $(X, M) \times (X_h, M_h)$ is

$$\mathcal{C}_h((u, p); (v_h, q_h)) = A(u, \Gamma_h v_h) + D(\Gamma_h v_h, p) + d(u, q_h) + G(p, q_h).$$

The next lemmas will heavily be used in the error analysis of problem (4.4) and (4.5). The mapping Γ_h satisfies the following properties [32]:

Lemma 4.1 *For each $K \in K_h$, if $v_h \in X_h$, $1 \le r \le \infty$, and $q > 0$ is an integer, then*

$$\int_K (v_h - \Gamma_h v_h) dx = 0, \ \|\Gamma_h v_h\|_0 \le c_7 \|v_h\|_0, \tag{4.6}$$

$$\|v_h - \Gamma_h v_h\|_{0,r,K} \le c_8 h_K \|v_h\|_{1,r,K}, \ \|v_h - \Gamma_h v_h\|_{0,r,\partial K} \le c_8 h_K^{q-1/r} \|v_h\|_{q,r,K}, \tag{4.7}$$

where h_K and ∂K are the diameter and boundary of the element K, respectively.

Obviously, the mapping Γ_h is a bridge between the finite volume methods and finite element methods. We can apply the existing results of finite element method to analyze the finite volume method. Unfortunately, there is still no optimal order error $O(h^2)$ between them. Thus, it is another major difficult to analyze the finite volume method for the nonlinear Navier–Stokes equations.

For the subsequent analysis, we now introduce a discrete analogue A_h of the Laplace operator A through the condition

$$(A_h u_h, v_h) = (\nabla u_h, \nabla v_h), \qquad u_h, v_h \in X_h.$$

Define

$$V_h = \{v_h \in X_h : d(v_h, q_h) = 0 \quad \forall\, q_h \in M_h\}.$$

The restriction of A_h to V_h is invertible, with the inverse A_h^{-1}. In addition, A_h is self-adjoint and positive definite. Therefore, we define the discrete Sobolev norm on V_h for any order $r \in R$ by

$$\|v_h\|_r = \|A_h^{r/2} v_h\|_0, \quad v_h \in V_h.$$

This discrete Laplace operator is firstly introduced in [22] to analyze and obtain optimal results for the complicated unsteady Navier–Stokes equations. In this paper, optimal and superconvergence results can be made by reasonable regularity and useful techniques because of complexity of finite volume method and the lower order $O(h)$ error between the test function of the finite volume method and that of finite element method.

Lemma 4.2 *The mapping Γ_h is self-adjoint with respect to the L^2-inner product:*

$$(u_h, \Gamma_h v_h) = (\Gamma_h u_h, v_h) \quad \forall u_h,\ v_h \in X_h. \tag{4.8}$$

In addition, the norm

$$|||u_h|||_0 = (u_h, \Gamma_h u_h)^{1/2}$$

is equivalent to the usual L^2-norm

$$c_9\|u_h\|_0 \le |||u_h|||_0 \le c_{10}\|u_h\|_0, \tag{4.9}$$

where the constants $c_9 > 0$ and $c_{10} > 0$ are independent of h. In particular, if $u_h(\cdot, t) \in X_h,\ v_h \in X_h,\ t \in [0, T]$, it holds

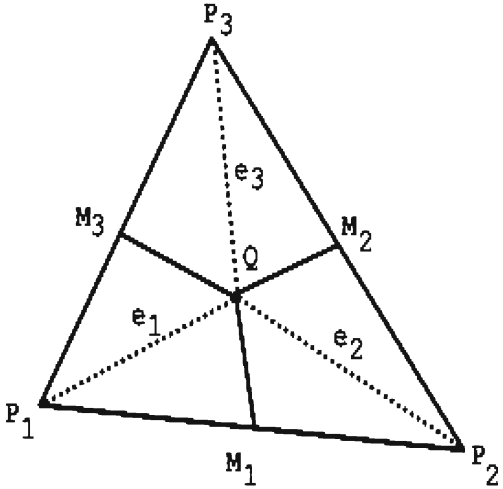
$$(u_{th}, \Gamma_h v_h) = (\Gamma_h u_{th}, v_h). \tag{4.10}$$

Finally, if $v_h = u_h$, it holds

$$(u_{th}, \Gamma_h u_h) = (\Gamma_h u_{th}, u_h) = \frac{1}{2}\frac{d}{dt}|||u_h|||_0^2. \tag{4.11}$$

Proof Results (4.8) and (4.9) can be found in [7]. For completeness, we prove (4.10). Denote the vertices of an element K by P_1, P_2, and P_3 (see Fig. 2). Let φ_i be the basis function in K and e_i, $i = 1, 2, 3$, be the quadrilateral

Fig. 2 A triangular element



$P_i M_i Q M_{i+2}$, $(M_5 = M_2,\ \ M_4 = M_1)$. For fixed t, any functions $u_h(\cdot, t)$ and $v_h(\cdot, t)$ have the unique representations

$$u_h(x,t)|_K = \sum_{i=1}^{3} u_i(t)\varphi_i(x), \ \ v_h(x,t)|_K = \sum_{j=1}^{3} v_j(t)\varphi_j(x), \ x \in \Omega, \ \varphi_i(x) \in X_h, \tag{4.12}$$

where $u_i(t)$ and $v_j(t)$, $i, j = 1, 2, 3$, are the values of u_h and v_h at the node P_i, respectively.

Then we see that

$$\begin{aligned}
(u_{th}, \Gamma_h v_h)_K &= \sum_{K\in K_h} \int_K \sum_{i=1}^{3} \frac{d}{dt} u_i(t)\varphi_i \Gamma_h v_h \, dx \\
&= \sum_{K\in K_h} \sum_{j=1}^{3} v_j(t) \int_{e_j} \sum_{i=1}^{3} \frac{d}{dt} u_i(t)\varphi_i \, dx \\
&= \sum_{K\in K_h} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{du_i(t)}{dt} v_j(t) \int_{e_j} \varphi_i \, dx \\
&= \sum_{K\in K_h} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{du_i(t)}{dt} v_j(t) \int_{e_i} \varphi_j \, dx \\
&= \sum_{K\in K_h} \sum_{i=1}^{3} \int_{e_i} \sum_{j=1}^{3} v_j(t)\varphi_j \Gamma_h \frac{du_i(t)}{dt} \, dx \\
&= (\Gamma_h u_{th}, v_h)_K,
\end{aligned} \tag{4.13}$$

where we used $\int_{e_i} \varphi_j dx = \int_{e_j} \varphi_i dx$ [7]. In particular, if $v_h = u_h$, we have

$$(u_{th}, \Gamma_h u_h) = (\Gamma_h u_{th}, u_h) = \frac{1}{2}\frac{d}{dt}|||u_h|||_0^2. \tag{4.14}$$

Consequently, (4.10) and (4.11) are proven. □

Lemma 4.3 [25, 40] *It holds that*

$$A(u_h, \Gamma_h v_h) = a(u_h, v_h) \qquad \forall u_h,\ v_h \in X_h, \tag{4.15}$$

with the following properties:

$$A(u_h, \Gamma_h v_h) = A(v_h, \Gamma_h u_h), \tag{4.16}$$

$$|A(u_h, \Gamma_h v_h)| \leq c\|u_h\|_1\|v_h\|_1, \tag{4.17}$$

$$|A(v_h, \Gamma_h v_h)| \geq c\|v_h\|_1^2. \tag{4.18}$$

Moreover, the bilinear form $D(\cdot, \cdot)$ satisfies

$$D(\Gamma_h v_h, q_h) = -d(v_h, q_h) \qquad \forall (v_h, q_h) \in (X_h, M_h). \tag{4.19}$$

Lemma 4.4 [25] *It holds that*

$$|\mathcal{C}_h((u_h, p_h); (v_h, q_h))| \leq c\,(\|u_h\|_1 + \|p_h\|_0)\,(\|v_h\|_1 + \|q_h\|_0)$$
$$\forall (u_h, p_h),\ (v_h, q_h) \in (X_h, M_h). \tag{4.20}$$

Moreover,

$$\sup_{(v_h,q_h)\in(X_h,M_h)} \frac{|\mathcal{C}_h((u_h, p_h); (v_h, q_h))|}{\|v_h\|_1 + \|q_h\|_0} \geq \beta\,(\|u_h\|_1 + \|p_h\|_0)$$
$$(u_h, p_h) \in (X_h, M_h), \tag{4.21}$$

where β is independent of h.

To derive error estimates for the finite volume solution (u_h, p_h), we define a projection operator $(R_h, Q_h):\ (X, M) \rightarrow (X_h, M_h)$ by

$$\mathcal{C}_h((u - R_h(u, p), p - Q_h(u, p)); (v_h, q_h)) = G(p, q_h) \forall (u, p) \in (X, M),$$
$$(v_h, q_h) \in (X_h, M_h), \tag{4.22}$$

which is well defined by Lemma 4.4. Also, it satisfies the following stability and approximation properties:

Lemma 4.5 *Under the assumption of* (A1), *the projection operator* (R_h, Q_h) *satisfies*

$$\|R_h(u, p)\|_1 + \|Q_h(u, p)\|_0 \leq c(\|u\|_1 + \|p\|_0), \tag{4.23}$$

$$\|u - R_h(u, p)\|_1 + \|p - Q_h(u, p)\|_0 \leq ch(\|u\|_2 + \|p\|_1), \tag{4.24}$$

for all $(u, p) \in (D(A), H^1(\Omega) \cap M)$.

Proof The stability property (4.23) can easily be shown by using Lemma 4.4. We focus on the proof of the approximation property (4.24). Setting $(I_h u - R_h(u,p), J_h p - Q_h(u,p)) = (e, \eta)$ and $E = u - I_h u$ in (4.22), we see that

$$\begin{aligned} A(e, \Gamma_h v_h) &+ D(\Gamma_h v_h, \eta) + d(e, q_h) + G(\eta, q_h) \\ &= -A(E, \Gamma_h v_h) - D(\Gamma_h v_h, p - J_h p) \\ &\quad -d(E, q_h) - G(p - J_h p, q_h) + G(p, q_h). \end{aligned} \tag{4.25}$$

Obviously, we deduce from (3.1), (3.2), (3.4) and (3.6) that

$$|d(E, q_h) + G(p - J_h p, q_h) - G(p, q_h)| \le ch(\|u\|_2 + \|p\|_1)\|q_h\|_0. \tag{4.26}$$

Then, using Green's formula, the Cauchy–Schwarz inequality, (3.2), (4.7) and Lemma 4.1, we have

$$\begin{aligned} |D(\Gamma_h v_h, p - J_h p)| &= \left| \sum_K (\nabla(p - J_h p), \Gamma_h v_h)_K \right| \\ &\le \left| \sum_K (\nabla(p - J_h p), \Gamma_h v_h - v_h)_K \right| + |(\nabla(p - J_h p), v_h)| \\ &\le \sum_K \|\nabla(p - J_h p)\|_{0,K} \|\Gamma_h v_h - v_h\|_{0,K} + |(\mathrm{div} v_h, p - J_h p)| \\ &\le ch\|p\|_1 \|v_h\|_1. \end{aligned} \tag{4.27}$$

Since

$$\begin{aligned} A(E, \Gamma_h v_h) &= -\sum_{j\in N_h} \sum_{K\cap \tilde{K}_j} \int_{\partial \tilde{K}_j \cap K} \frac{\partial E}{\partial n} \Gamma_h v_h ds \\ &= \sum_K \int_{\partial K} \frac{\partial E}{\partial n} \Gamma_h v_h ds - \sum_K (\Delta E, \Gamma_h v_h)_K \\ &= \sum_K \int_{\partial K} \left(\frac{\partial E}{\partial n} - \frac{\partial \Pi_h E}{\partial n} \right) (\Gamma_h v_h - v_h) ds - \sum_K (\Delta E, \Gamma_h v_h - v_h)_K \\ &\quad - \sum_K \int_{\partial K} \frac{\partial E}{\partial n} v_h ds + \sum_K (\nabla E, \nabla v_h)_K, \end{aligned}$$

a similar argument yields

$$\begin{aligned} |A(E, \Gamma_h v_h)| &\le \sum_K \|\nabla(E - \Pi_h E)\|_{0,\partial K} \|\Gamma_h v_h - v_h\|_{0,\partial K} \\ &\quad + \sum_K \|E\|_{2,K} \|v_h - \Gamma_h v_h\|_{0,K} + c\|E\|_1 \|v_h\|_1 \\ &\le ch\|u\|_2 \|v_h\|_1. \end{aligned} \tag{4.28}$$

Using all the inequalities (4.26)–(4.28) in (4.25), we find

$$\|e\|_1 + \|\eta\|_0 \le \beta^{-1} \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{|\mathcal{C}_h((e,\eta);(v_h,q_h))|}{\|v_h\|_1 + \|q_h\|_0}$$
$$\le ch(\|u\|_2 + \|p\|_1). \tag{4.29}$$

Finally, combining (4.29), (3.1), and (3.2) gives (4.24). □

So far, we refer to two kinds of the Stokes projections, based on Galerkin system and Petrov–Galerkin system. The first Stokes projection defined in (3.13) has optimal convergence results and leads to the optimal error estimate for the finite element discretization of the transient Navier–Stokes equations. As for the second one defined in (4.22), we can not obtain the optimal error estimate of $\|u - R_h(u,p)\|_0$ because of lower order error between test function of the finite element method and finite volume method. However, it is worth to mention that both projects complement perfectly each other to obtain the optimal results for the finite volume method of the transient Navier–Stokes equations without any additional regularity on the exact solution.

Then, we will prove several useful estimates for the trilinear terms.

Lemma 4.6 *It holds that, for $u_h,\ v_h,\ w_h \in X_h$,*

$$|b(u_h, v_h, \Gamma_h w_h - w_h)|$$
$$\le ch\|u_h\|_1\|v_h\|_1\|w_h\|_1, \tag{4.30}$$
$$|b(u_h, v - v_h, \Gamma_h w_h - w_h) + b(v - v_h, u_h, \Gamma_h w_h - w_h)|$$
$$\le ch^{1/2}\|u_h\|_0^{1/2}\|u_h\|_1^{1/2}\|v - v_h\|_1\|w_h\|_1, \tag{4.31}$$
$$|b(u, v - v_h, \Gamma_h w_h - w_h) + b(v - v_h, u, \Gamma_h w_h - w_h)|$$
$$\le c\|u\|_2\|v - v_h\|_1\|\Gamma_h w_h - w_h\|_0. \tag{4.32}$$

Proof Using (2.3), (3.3), (3.6) and Lemma 4.1, we see that

$$|b(u_h, v_h, \Gamma_h w_h - w_h)|$$
$$= \left(((u_h - \Pi_h u_h)\cdot\nabla)v_h + \frac{1}{2}\mathrm{div}\, u_h(v_h - \Pi_h v_h), \Gamma_h w_h - w_h\right)$$
$$\le c(\|u_h - \Pi_h u_h\|_0\|\nabla v_h\|_{L^\infty} + \|v_h - \Pi_h v_h\|_0\|\nabla u_h\|_{L^\infty})\|\Gamma_h w_h - w_h\|_0$$
$$\le ch^2\left(\|u_h\|_1\|A_h^{1/2}v_h\|_0^{1/2}\|A_h^{3/2}v_h\|_0^{1/2} + \|v_h\|_1\|A_h^{1/2}u_h\|_0^{1/2}\|A_h^{3/2}u_h\|_0^{1/2}\right)\|w_h\|_1$$
$$\le ch\|u_h\|_1\|v_h\|_1\|w_h\|_1.$$

Setting $e = v - v_h$, we deduce from (2.3), (3.3), (3.6), and Lemma 4.1, that

$$
\begin{aligned}
&|b(u_h, v - v_h, \Gamma_h w_h - w_h)| \\
&\quad = \left((u_h \cdot \nabla) e + \frac{1}{2} \mathrm{div}\, u_h (e - \Pi_h e), \Gamma_h w_h - w_h \right) \\
&\quad \leq (\|u_h\|_{L^\infty} \|e\|_1 + \|\nabla u_h\|_{L^\infty} \|e - \Pi_h e\|_0) \|\Gamma_h w_h - w_h\|_0 \\
&\quad \leq ch \left(\|u_h\|_0^{1/2} \|A_h u_h\|_0^{1/2} \|e\|_1 + \|A_h^{1/2} u_h\|_0^{1/2} \|A_h^{3/2} u_h\|_0^{1/2} \|e - \Pi_h e\|_0 \right) \|w_h\|_1 \\
&\quad \leq ch^{1/2} \|u_h\|_0^{1/2} \|u_h\|_1^{1/2} \|v - v_h\|_1 \|w_h\|_1.
\end{aligned}
$$

Similarly, we can obtain estimate of the second trilinear terms of (4.31) by the same approach. Also, by (2.3) and the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
&|b(u, v - v_h, \Gamma_h w_h - w_h) + b(v - v_h, u, \Gamma_h w_h - w_h)| \\
&\quad \leq c(\|u\|_{L^\infty} \|v - v_h\|_1 + \|\nabla u\|_{L^4} \|v - v_h\|_{L^4}) \|\Gamma_h w_h - w_h\|_0 \\
&\quad \leq c\|u\|_2 \|v - v_h\|_1 \|\Gamma_h w_h - w_h\|_0.
\end{aligned}
$$

As a result, (4.30), (4.31) and (4.32) are shown. □

5 Stability and error analysis

In this paper, the main focus is to analyze the stabilized finite volume method based on the relationship between the finite element method and the finite volume method and some additional analytical techniques. As noted earlier, there are several difficulties in analyzing this finite volume method for the transient Navier–Stokes equations. Some remarks need be made. First, the analysis requires a slightly extra regularity on the source force to obtain the optimal L^2-norm for the velocity. The counterexample in [14, 23] showed that the finite volume solutions approximated by the conforming linear elements cannot have the optimal L^2-norm convergence rate if the exact solution is in $H^2(\Omega)$ and the source term is only in $L^2(\Omega)$ for a saddle point problem. Second, additional attention is here required to treat the trilinear term for the nonlinear Naiver-Stokes equations because of its losing the anti-symmetric property. Third, additional techniques need be provided to analyze the parabolic system that is in the form of the Petrov–Galerkin system generated by two different Stokes projections.

From the point of view of implementation, the stabilized finite volume method consists of several subroutines for solving the transient Navier–Stokes equations. First, we solve the Stokes equations approximated by the lowest equal-order finite element pair to obtain an initial value for the iterative finite volume method for the transient Navier–Stokes equations. Then two nested loops in the algorithm are involved in time and space for solving these equations. In addition, a nested loop in space is contained in the loop of time by the Picard iterative finite volume method at each fixed time step.

Finite volume algorithm for the transient Navier–Stokes equations

Step I Find $(u_h^0, p_h^0) \in (X_h, M_h)$ satisfying the following **Stokes** equations:

$$\mathcal{C}_h((u_h^0, p_h^0); (v_h, q_h)) = (f, \Gamma_h v_h) \quad \forall (v_h, q_h) \in (X_h, M_h).$$

Moreover, set the iterative step $m = 0, 1, 2, \ldots$, the error of two successive solutions $e_m = \sqrt{(u_h^m - u_h^{m-1})^2 + (p_h^m - p_h^{m-1})^2}$ $(m = 0,\ e_m = 0)$, and a sufficient small iterative tolerance $\epsilon > 0$.

Step II Solve the stationary linear Naiver-Stokes equations (**LNS**) at each fixed step by applying the Picard iterative finite volume approximation:

$$\left(\left(u_h^m - u_h^{m-1}\right)/\tau, \Gamma_h v_h\right) + \mathcal{C}_h\left(\left(u_h^m, p_h^m\right);\right.$$
$$\left.(v_h, q_h)\right) + b\left(u_h^{m-1}, u_h^m, \Gamma_h v_h\right) = (f, \Gamma_h v_h).$$

Routine:
$[u_h^0, p_h^0]$=**Stokes**$(\tilde{K}_h,\ f)$;
for $i = 0, 1, 2, \cdots T/dt$;
while $(e_m > \epsilon)$ **do**
$(u_h^m, p_h^m) \to (u_h^{m-1}, p_h^{m-1})$;
$[u_h^m, p_h^m]$=**LNS**$(\tilde{K}_h, u_h^{m-1}, p_h^{m-1}, f)$;
end while
end for

We now prove that the system in (4.4) and (4.5) is solvable at each fixed time. As an example, we prove a stability result in the next lemma.

Lemma 5.1 *Under the assumptions of* (A1) *and* (A2)*, there is the parameter h_0 such that*

$$0 < h_0(h) = \frac{2c_4c_6c_8\kappa_0|\log h|^{1/2}h^2}{\nu} < 1/2 \tag{5.1}$$

for sufficiently small $h > 0$, it holds that, for $t \in [0, T]$,

$$\|u_h(t)\|_0^2 + \textstyle\int_0^t \left(\nu\|u_h\|_1^2 + G(p_h, p_h)\right) ds \le c, \tag{5.2}$$

$$\|u(t) - u_h(t)\|_0^2 + \textstyle\int_0^t \left(\nu\|u - u_h\|_1^2 + G(p - p_h, p - p_h)\right) ds \le ch^2. \tag{5.3}$$

Furthermore, we have

$$\nu\|u_h(t)\|_1^2 + G(p_h(t), p_h(t)) + \int_0^t \|u_{ht}\|_0^2 ds \le c. \tag{5.4}$$

Proof Based on the previous finite volume algorithm, the initial value produced by the finite volume solution of the Stokes equations can obviously be bounded by some positive constant independent of h [25]. At a fixed time step, we consider the iterative finite volume scheme described above:

$$(u_{th}, \Gamma_h v_h) + \mathcal{C}_h((u_h, p_h); (v_h, q_h)) + b(\bar{v}_h, u_h, \Gamma_h v_h) = (f, \Gamma_h v_h). \tag{5.5}$$

Then we assume that

$$\nu\|\bar{v}_h\|_1^2 + G(q_h, q_h) + \int_0^t \|\bar{v}_{ht}\|_0^2 ds \le c.$$

Taking $(v_h, q_h) = (u_h, p_h)$ in (5.5) and using Lemma 4.2 and (2.7), we see that

$$\frac{1}{2}\frac{d}{dt}\||u_h\||_0^2 + \nu\|u_h\|_1^2 + G(p_h, p_h) + b(\bar{v}_h, u_h, \Gamma_h u_h - u_h) = (f, \Gamma_h u_h). \quad (5.6)$$

Using (2.3), (3.3), (3.6), and (4.7), we have

$$\begin{aligned}
&|b(\bar{v}_h, u_h, \Gamma_h u_h - u_h)| \\
&\quad = \left(((\bar{v}_h - \Pi_h \bar{v}_h)\cdot\nabla)u_h + \frac{1}{2}\text{div }\bar{v}_h(u_h - \Pi_h u_h), \Gamma_h u_h - u_h\right) \\
&\quad \le (\|\bar{v}_h - \Pi_h\bar{v}_h\|_0\|\nabla u_h\|_{L^\infty} + \|u_h - \Pi_h u_h\|_0\|\nabla\ \bar{v}_h\|_{L^\infty})\|\Gamma_h u_h - u_h\|_0 \\
&\quad \le 2c_4c_6c_8|\log h|^{1/2}h^2\|\bar{v}_h\|_1\|u_h\|_1^2
\end{aligned}$$

and

$$|(f, \Gamma_h u_h)| \le c_7\|f\|_0\|u_h\|_0 \le \frac{1}{4}c_7^2\|f\|_0^2 + \|u_h\|_0^2.$$

Then we deduce from (5.5) and (5.6) that

$$\frac{1}{2}\frac{d}{dt}\||u_h\||_0^2 + \nu(1-h_0)\|u_h\|_1^2 + G(p_h, p_h) \le c_7^2\|f\|_0^2 + \frac{1}{4}\|u_h\|_0^2.$$

Integrating the above inequality from 0 to t with respect to time, applying the Gronwall inequality, and noting that

$$\||u_h(0)\||_0 \le \||R_h(u_0, p_0)\||_0 \le c(\|u_0\|_1 + \|p_0\|_0) \le c_{11},$$

we obtain

$$\|u_h(t)\|_0^2 + \int_0^t(\nu\|u_h\|_1^2 + G(p_h, p_h))ds \le c_9^{-1}\left(2c_7^2\int_0^t \|f\|_0^2\, ds + c_{11}^2\right)e^{T/2},$$

which implies the desired result (5.2).

Multiplying (2.1) and (2.2) by $\Gamma_h v_h$ and q_h, integrating over $\tilde{K}$ and K, respectively, and using (4.4) and (4.5), we see that

$$\begin{aligned}
(u_t - u_{ht}, \Gamma_h v_h) + \mathcal{C}_h((u-u_h, p-p_h); (v_h, q_h)) + b(u-u_h, u, v_h)& \\
+ b(u_h, u-u_h, v_h) + b(u-u_h, u, \Gamma_h v_h - v_h)& \\
+ b(u_h, u-u_h, \Gamma_h v_h - v_h) = G(p, q_h).& \quad (5.7)
\end{aligned}$$

Setting $(e, \eta) = (\bar{R}_h(u, p) - u_h, \bar{Q}_h(u, p) - p_h)$ and $E = u - \bar{R}_h(u, p)$, using Lemma 4.2 and (2.7), and taking $(v_h, q_h) = (e, \eta)$ in (5.7), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|||e|||_0^2 &+ \nu\|e\|_1^2 + G(\eta,\eta) + b(E+e, u, e) + b(u_h, E, e)\\ &+ b(E+e, u, \Gamma_h e - e) + b(u_h, E+e, \Gamma_h e - e) = -(E_t, \Gamma_h e)\\ &- A(E, \Gamma_h e) - D(\Gamma_h e, p - \bar{Q}_h(u, p))\\ &- d(u - \bar{R}_h(u, p), \eta) - G(p - \bar{Q}_h(u, p), \eta) + G(p, \eta). \qquad (5.8)\end{aligned}$$

Using Lemma 2.2, (2.3), (2.7)–(2.9), and the Young inequality, we see that

$$\begin{aligned}|(E_t, \Gamma_h e)| &\le c\|E_t\|_0\|e\|_0 \le ch^2\left(\|u_t\|_1^2 + \|p_t\|_0^2\right) + \|e\|_0^2,\\ |b(E, u, e)| &\le c\|E\|_1\|u\|_1\|e\|_1 \le \frac{\nu}{16}\|e\|_1^2 + c\|E\|_1^2\|u\|_1^2,\\ |b(e, u, e)| &\le c\left\{\|e\|_0\|e\|_1\|u\|_1 + \|e\|_0^{1/2}\|e\|_1^{3/2}\|u\|_0^{1/2}\|u\|_1^{1/2}\right\}\\ &\le \frac{\nu}{16}\|e\|_1^2 + c\left(1+\|u\|_0^2\right)\|u\|_1^2\|e\|_0^2,\\ |b(u_h, E, e)| &\le c\|u_h\|_1\|E\|_1\|e\|_1 \le \frac{\nu}{16}\|e\|_1^2 + c\|u_h\|_1^2\|E\|_1^2.\end{aligned}$$

Also, using (2.3), Lemma 4.6 and (5.2), an inverse inequality, and the Schwarz inequality, we obtain

$$\begin{aligned}|b(E, u, \Gamma_h e - e)| &\le c\|E\|_1\|u\|_2\|\Gamma_h e - e\|_0 \le \frac{\nu}{16}\|e\|_1^2 + ch^2\|u\|_2^2\|E\|_1^2,\\ |b(e, u, \Gamma_h e - e)| &\le c\|u\|_2\|e\|_1\|\Gamma_h e - e\|_0 \le \frac{\nu}{16}\|e\|_1^2 + c\|u\|_2^2\|e\|_0^2,\\ |b(u_h, e, \Gamma_h e - e)| &\le ch\|u_h\|_1\|e\|_1\|e\|_1 \le \frac{\nu}{16}\|e\|_1^2 + c\|u_h\|_1^2\|e\|_0^2,\\ |b(u_h, E, \Gamma_h e - e)| &\le ch^{1/2}\|u_h\|_0^{1/2}\|u_h\|_1^{1/2}\|E\|_1\|e\|_1 \le \frac{\nu}{16}\|e\|_1^2 + c\|E\|_1^2.\end{aligned}$$

Moreover, we deduce from (2.1), (2.2), (3.13), (4.4), (4.5), (4.15) and (4.19) that

$$\begin{aligned}|A(E, \Gamma e) + D(\Gamma_h e, p - \bar{Q}_h(u, p))| &= (f - u_t - (u\cdot\nabla)u, \Gamma_h e - e)\\ &\le ch(\|f\|_0 + \|u_t\|_0 + \|u\|_2\|u\|_1)\|e\|_1\\ &\le \frac{\nu}{16}\|e\|_1^2 + ch^2\left(\|f\|_0^2 + \|u_t\|_0^2\right),\\ |d(E,\eta) + G(p - \bar{Q}_h(u, p), \eta) - G(p, \eta)| &= 0. \qquad (5.9)\end{aligned}$$

Now, substituting these inequalities into (5.8) and using (A2), we find

$$\begin{aligned}\frac{d}{dt}|||e|||_0^2 + \nu\|e\|_1^2 + G(\eta,\eta) \le c\Big\{&h^2\left(\|u_t\|_0^2 + \|f\|_0^2 + \|u\|_2\|u\|_1\right)\\ &+ \left(1 + \|u\|_2^2 + \|u_h\|_1^2\right)\|u\|_1^2\|e\|_0^2 + \left(1 + \|u_h\|_1^2 + \|u\|_2^2\right)\|E\|_1^2\Big\}.\end{aligned}$$

Therefore, integrating the above inequality from 0 to t, we deduce from the Schwarz inequality, Lemmas 2.1 and 2.2, (5.2), and (4.9) that

$$\|e(t)\|_0^2 + \int_0^t \left(\nu\|e\|_1^2 + G(\eta,\eta)\right) ds \le ch^2, \tag{5.10}$$

which, together with Lemmas 2.2 and 3.2, yields

$$\|u(t) - u_h(t)\|_0^2 + \int_0^t \left(\nu\|u - u_h\|_1^2 + G(p - p_h, p - p_h)\right) ds \le ch^2. \tag{5.11}$$

Thus the desired result (5.3) follows.

Differentiating the term $d(u_h, q_h) + G(p_h, q_h)$ with respect to time t, taking $(v_h, q_h) = (u_{ht}, p_h)$ in (4.4) and (4.5), and using Lemma 4.3, we see that

$$\begin{aligned} &|||u_{ht}|||_0^2 + \frac{1}{2}\frac{d}{dt}(\nu\|u_h\|_1^2 + G(p_h, p_h)) + b\,(u_h, u - u_h, u_{ht} - \Gamma_h u_{ht}) \\ &\quad - b\,(u - u_h, u, \Gamma_h u_{ht}) - b\,(u_h, u - u_h, u_{ht}) + b\,(u, u, \Gamma_h u_{ht}) = (f, \Gamma_h u_{ht}). \end{aligned} \tag{5.12}$$

Applying (2.3), (2.7)–(2.9), (3.3), Lemma 4.6 and the Young inequality, leads to

$$\begin{aligned} |b\,(u_h, u - u_h, u_{ht} - \Gamma_h u_{ht})| &\le ch^{1/2}\|u_h\|_0^{1/2}\|u_h\|_1^{1/2}\|u - u_h\|_1\|u_{ht}\|_1 \\ &\le \frac{1}{8}\|u_{ht}\|_0^2 + ch^{-2}\|u_h\|_0^2\|u - u_h\|_1^2, \\ |b\,(u - u_h, u, \Gamma_h u_{ht})| &\le c\|u - u_h\|_1\|u\|_2\|\Gamma_h u_{ht}\|_0 \\ &\le \frac{1}{8}\|u_{ht}\|_0^2 + c\|u\|_2^2\|u - u_h\|_1^2, \end{aligned}$$

$$\begin{aligned} |b\,(u, u, \Gamma_h u_{ht} - u_{ht}))| &\le c\|u\|_2\|u\|_1\|u_{ht}\|_0 \\ &\le \frac{1}{8}\|u_{ht}\|_0^2 + c\|u\|_1^2\|u\|_2^2, \end{aligned}$$

The combination of the inverse inequality (3.3) and the Cauchy–Schwarz inequality implies that

$$\begin{aligned} &|b\,(u_h, u - u_h, u_{ht}))| \\ &\quad \le ch^{-1/2}\left(\|u_h\|_0^{1/2}\|u_h\|_1^{1/2}\|u - u_h\|_1 + \|u_h\|_1\|u - u_h\|_0^{1/2}\|u - u_h\|_1^{1/2}\right)\|u_{ht}\|_0 \\ &\quad \le \frac{1}{8}\|u_{ht}\|_0^2 + ch^{-2}\|u_h\|_0\left(\|u_h\|_0\|u - u_h\|_1^2 + \|u_h\|_1\|u - u_h\|_0\|u - u_h\|_1\right). \end{aligned}$$

In addition, a simple computation shows that

$$|(f, \Gamma_h u_{ht})| \le \|f\|_0\|u_{ht}\|_0 \le \frac{1}{4}\|u_{ht}\|_0^2 + c\|f\|_0^2.$$

Combining all these inequalities with (4.2) and (5.2) yields

$$\|u_{ht}\|_0^2 + \frac{d}{dt}(\nu\|u_h\|_1^2 + G(p_h, p_h)) \leq c\left\{\|u\|_2^2\|u-u_h\|_1^2 + \|u\|_1^2\|u\|_2^2 + \|f\|_0^2 + h^{-2}\|u_h\|_0\left(\|u_h\|_0\|u-u_h\|_1^2 + \|u_h\|_1\|u-u_h\|_0\|u-u_h\|_1\right)\right\}. \quad (5.13)$$

Then, integrating (5.13) from 0 to t with respect to time, using Lemma 2.2 and (4.23), and noting that

$$\nu\|u_h(0)\|_1^2 + G(p_h(0), p_h(0)) \leq \|\bar{R}_h(u(0), p(0))\|_1^2 + \|\bar{Q}_h(u(0), p(0))\|_0^2 \leq c\left(\|u\|_1^2 + \|p\|_0^2\right),$$

we deduce from (5.2), (5.3), (A2), and Lemma 2.2 that

$$\nu\|u_h(t)\|_1^2 + G(p_h(t), p_h(t)) + \int_0^t \|u_{ht}\|_0^2 ds \leq c\left(\|u\|_1^2 + \|p\|_0^2\right) + c\left\{\|u\|_2^2\int_0^t \|u-u_h\|_1^2 ds + \int_0^t \left(\|u\|_2^2 + \|f\|_0^2\right) ds + h^{-2}\int_0^t \|u-u_h\|_1^2 ds + h^{-2}\|u-u_h\|_0\left(\int_0^t \|u_h\|_1^2 ds\right)^{1/2}\left(\int_0^t \|u-u_h\|_0^2 ds\right)^{1/2}\right\} \leq c.$$

By using a mathematical induction argument, we complete the proof of (5.4). □

Lemma 5.2 *Under the assumptions of* (A1)–(A2)*, it holds that, for* $t \in [0, T]$,

$$\|u_{ht}(t)\|_0^2 + \int_0^t \left(\nu\|u_{ht}\|_1^2 + G(p_{ht}, p_{ht})\right) ds \leq c, \quad (5.14)$$

$$\tau(t)\left(\nu\|u_{ht}(t)\|_1^2 + G(p_{ht}(t), p_{ht}(t))\right) + \int_0^t \tau(s)\|u_{htt}\|_0^2\, ds \leq c. \quad (5.15)$$

Proof By differentiating (4.4) and (4.5) with respect to time, it follows that

$$(u_{htt}, \Gamma_h v_h) + \mathcal{C}_h((u_{ht}, p_{ht}); (v_h, q_h)) + b(u_{ht}, u_h, \Gamma_h v_h) + b(u_h, u_{ht}, \Gamma_h v_h) = (f_t, \Gamma_h v_h) \quad \forall (v_h, q_h) \in (X_h, M_h). \quad (5.16)$$

Taking $(v_h, q_h) = 2(u_{ht}, p_{ht})$ in (5.16) and using Lemma 4.2, we have

$$\frac{d}{dt}|||u_{ht}|||_0^2 + 2\nu\|u_{ht}\|_1^2 + 2G(p_{ht}, p_{ht}) + 2b(u_{ht}, u_h, \Gamma_h u_{ht} - u_{ht}) + 2b(u_{ht}, u_h, u_{ht}) + 2b(u_h, u_{ht}, \Gamma_h u_{ht} - u_{ht}) \leq \|u_{ht}\|_0^2 + c\|f_t\|_0^2. \quad (5.17)$$

Using (2.3) and (4.30), we see that

$$\begin{aligned}2|b(u_{ht}, u_h, u_{ht})| &\le c\left(\|u_{ht}\|_0\|u_h\|_1\|u_{ht}\|_1 + \|u_h\|_0^{1/2}\|u_h\|_1^{1/2}\|u_{ht}\|_0^{1/2}\|u_{ht}\|_1^{3/2}\right)\\ &\le \frac{\nu}{2}\|u_{ht}\|_1^2 + c\|u_h\|_0^2\|u_h\|_1^2\|u_{ht}\|_0^2,\\ 2|b(u_{ht}, u_h, \Gamma_h u_{ht} - u_{ht}) + b(u_h, u_{ht}, \Gamma_h u_{ht} - u_{ht})| &\le ch\|u_h\|_1\|u_{ht}\|_1\|u_{ht}\|_1\\ &\le \frac{\nu}{2}\|u_{ht}\|_1^2 + c\|u_h\|_1^2\|u_{ht}\|_0^2,\end{aligned}$$

which, together with (5.17), gives

$$\frac{d}{dt}\||u_{ht}\||_0^2 + \nu\|u_{ht}\|_1^2 + G(p_{ht}, p_{ht}) \le c\left\{(1+\|u_h\|_0^2)\|u_h\|_1^2\|u_{ht}\|_0^2 + \|f_t\|_0^2\right\}. \tag{5.18}$$

Integrating (5.18) with respect to time and using Lemmas 2.1, 4.2, 5.1 and (A2), we obtain (5.14).

Now, differentiating again the term $d(u_{ht}, q_h) + G(p_{ht}, q_h)$ in (4.4), and taking $(v_h, q_h) = (u_{htt}, p_{ht})$, we see that

$$\begin{aligned}&\||u_{htt}\||_0^2 + \frac{1}{2}\frac{d}{dt}\left(\nu\|u_{ht}\|_1^2 + G(p_{ht}, p_{ht})\right) + b(u, u_{ht}, \Gamma_h u_{htt}) + b(u_{ht}, u, \Gamma_h u_{htt})\\ &\quad + b(u_{ht}, u_h - u, \Gamma_h u_{htt}) + b(u_h - u, u_{ht}, \Gamma_h u_{htt}) \le \frac{1}{8}\|u_{htt}\|_0^2 + c\|f_t\|_0^2.\end{aligned} \tag{5.19}$$

Obviously, using (2.9), (3.3), and Lemma 4.6, we have

$$\begin{aligned}&|b(u, u_{ht}, \Gamma_h u_{htt}) + b(u_{ht}, u, \Gamma_h u_{htt})|\\ &\quad \le c\|u\|_2\|u_{ht}\|_1\|u_{htt}\|_0 \le \frac{1}{4}\|u_{htt}\|_0^2 + c\|u\|_2^2\|u_{ht}\|_1^2,\\ &|b(u_{ht}, u_h - u, \Gamma_h u_{htt} - u_{htt}) + b(u_h - u, u_{ht}, \Gamma_h u_{htt} - u_{htt})|\\ &\quad \le ch^{1/2}\|u_{ht}\|_0^{1/2}\|u_{ht}\|_1^{1/2}\|u - u_h\|_1\|u_{htt}\|_1\\ &\quad \le \frac{1}{8}\|u_{htt}\|_0^2 + ch^{-2}\|u_{ht}\|_0^2\|u - u_h\|_1^2.\end{aligned}$$

Combining these estimates with (5.19), it follows from Lemmas 2.2, 4.2 and (3.3) that

$$\begin{aligned}&\|u_{htt}\|_0^2 + \frac{d}{dt}\left(\nu\|u_{ht}\|_1^2 + G(p_{ht}, p_{ht})\right)\\ &\quad \le c\left\{\|u\|_2^2\|u_{ht}\|_1^2 + h^{-2}\|u_{ht}\|_0^2\|u - u_h\|_1^2 + \|f_t\|_0^2\right\}.\end{aligned} \tag{5.20}$$

Finally, multiplying (5.20) by $\tau(t)$, integrating from 0 to t, and using Lemma 2.2, we obtain

$$\begin{aligned}
&\tau(t)\left(\nu\|u_{ht}(t)\|_1^2+G(p_{ht}(t),p_{ht}(t))\right)+\int_0^t\tau(s)\|u_{htt}\|_0^2ds\\
&\quad\leq\left(c\left\{\|u\|_2^2\int_0^t\|u_{ht}\|_1^2ds+h^{-2}\|u_{ht}\|_0^2\int_0^t\|u-u_h\|_1^2ds+\int_0^t\|f_t\|_0^2ds\right\}\right.\\
&\qquad\left.+\int_0^t\nu\|u_{ht}\|_1^2+G(p_{ht},p_{ht})\right)ds. \qquad (5.21)
\end{aligned}$$

Therefore, combining (5.21) with Lemmas 2.2, (5.3), (5.14) and (A2) completes the proof of (5.15). □

Lemma 5.3 *Under the assumptions of* (A1)–(A2)*, it holds that, for* $t\in[0,T]$,

$$\nu\|u(t)-u_h(t)\|_1^2+\int_0^t\|u_t-u_{ht}\|_0^2\,ds\leq ch^2. \qquad (5.22)$$

Proof Differentiating the term $d(u-u_h,q_h)+G(p-p_h,q_h)$, taking $(v_h,q_h)=(e_t,\eta)=(\bar{R}_{ht}(u,p)-u_{ht},\bar{Q}_h(u,p)-p_h)$ in (5.7), and using Lemma 4.2, we see that

$$\begin{aligned}
&\|e_t\|_0^2+\frac{1}{2}\frac{d}{dt}\left(\nu\|e\|_1^2+G(\eta,\eta)\right)+b(u-u_h,u,e_t)+b(u,u-u_h,e_t)\\
&\quad-b(u-u_h,u-u_h,e_t)+b(u,u-u_h,\Gamma_he_t-e_t)+b(u-u_h,u,\Gamma_he_t-e_t)\\
&\quad-b(u-u_h,u-u_h,\Gamma_he_t-e_t)=-(E_t,\Gamma_he_t)-A(E,\Gamma_he_t)\\
&\quad-D(\Gamma_he_t,p-\bar{Q}_h(u,p))-d(E_t,\eta)-G(p_t-\bar{Q}_{ht}(u,p),\eta)+G(p_t,\eta).
\end{aligned} \qquad (5.23)$$

Applying (2.3), (2.7)–(2.9), (3.3), and Lemmas 2.2, 4.1, we have

$$\begin{aligned}
|b(u-u_h,u,\Gamma_he_t)+b(u,u-u_h,e_t)|&\leq c\|u\|_2\|u-u_h\|_1\|e_t\|_0\\
&\leq\frac{1}{12}\|e_t\|_0^2+c\|u\|_2^2\|u-u_h\|_1^2,\\
|b(u_h-u,u-u_h,e_t)|&\leq c\|u-u_h\|_0^{1/2}\|u-u_h\|_1^{3/2}\|e_t\|_0^{1/2}\|e_t\|_1^{1/2}\\
&\leq\frac{1}{12}\|e_t\|_0^2+ch^{-1}\|u-u_h\|_0\|u-u_h\|_1^3,
\end{aligned}$$

Due to (2.3), (3.3), (4.32), (5.3) and the Cauchy–Schwarz inequality, we bound the following two inequalities

$$
\begin{aligned}
&|b(u, u-u_h, \Gamma_h e_t - e_t) + b(u-u_h, u, \Gamma_h e_t - e_t)| \\
&\quad \le c\|u\|_2\|u-u_h\|_1\|e_t\|_0 \\
&\quad \le \frac{1}{12}\|e_t\|_0^2 + c\|u\|_2^2\|u-u_h\|_1^2, \\
&|b(u-u_h, u-u_h, e_t - \Gamma_h e_t)| \\
&\quad \le c\|u-u_h\|_0^{1/2}\|u-u_h\|_1^{3/2}\|\Gamma_h e_t - e_t\|_{L^4} \\
&\quad \le ch\|u-u_h\|_0^{1/2}\|u-u_h\|_1^{3/2}\|A_h^{1/2} e_t\|_{L^4} \\
&\quad \le ch^{1/2}\|u-u_h\|_1^{3/2}\|e_t\|_0 \\
&\quad \le \frac{1}{12}\|e_t\|_0^2 + c(\|u_h\|_1 + \|u\|_1)\|u-u_h\|_1^2.
\end{aligned}
$$

Using Lemma 3.2 and the Young inequality, leads to

$$
|(E_t, \Gamma_h e_t)| \le c\|E_t\|_0\|e_t\|_0 \le ch^2\left(\|u_t\|_1^2 + \|p_t\|_0^2\right) + \frac{1}{12}\|e_t\|_0^2.
$$

Moreover, using (3.4)–(3.6), (3.13) and the same approach as (5.9) gives

$$
\begin{aligned}
&|A(E, \Gamma_h e_t) + D(\Gamma_h e_t, p_t - \bar{Q}_{ht}(u,p))| \\
&\quad = |(f - (u\cdot\nabla)u - u_t - \Pi_h(f - (u\cdot\nabla)u - u_t), \Gamma_h e_t - e_t)| \\
&\quad \le ch^2\left(\|f\|_1^2 + \|u_t\|_1^2\right) + \frac{1}{12}\|e_t\|_0^2, \\
&|d(E_t, \eta) + G(p_t - \bar{Q}_{ht}(u,p), \eta) - G(p_t, \eta)| = 0.
\end{aligned}
$$

Hence combining these inequalities with (5.23) yields

$$
\begin{aligned}
&\|e_t\|_0^2 + \frac{d}{dt}\left(\nu\|e\|_1^2 + G(\eta,\eta)\right) \\
&\quad \le c\Big\{\left(\|u\|_2^2 + \|u_h\|_1^2\right)\|u-u_h\|_1^2 + h^2\left(\|u_t\|_1^2 + \|f\|_1^2\right) \\
&\qquad + ch^{-1}\|u-u_h\|_0(\|u_h\|_1 + \|u\|_1)\|u-u_h\|_1^2\Big\}.
\end{aligned}
$$

Integrating the above inequality from 0 to t and using (5.3) and (5.4) and (A2), we can see that

$$
\begin{aligned}
&\nu\|e(t)\|_1^2 + G(\eta(t), \eta(t)) + \int_0^t \|e_t\|_0^2 ds \\
&\quad \le ch^2\int_0^t\left(\|u_t\|_1^2 + \|f\|_1^2\right) ds + ch^2\left(\|u\|_1^2 + \|u_h\|_1^2\right) \le ch^2, \quad (5.24)
\end{aligned}
$$

which, together with Lemmas 2.2 and 3.2, yields the desired result (5.22). □

Lemma 5.4 *Under the assumptions of* (A1)–(A2)*, it holds that, for* $t \in [0, T]$,

$$\tau(t)\|u_t(t) - u_{ht}(t)\|_0^2 + \nu \int_0^t \tau(s)\|e_t\|_1^2 ds \le ch^2. \tag{5.25}$$

Proof Multiplying (2.1) and (2.2) by $\Gamma_h v_h$ and q_h, integrating over $\tilde{K}$ and K, respectively, and using (4.4) and (4.5), yields that

$$\begin{aligned}(u_t - u_{ht}, \Gamma_h v_h) &+ \mathcal{C}_h((u - u_h, p - p_h); (v_h, q_h)) + b(u - u_h, u, \Gamma_h v_h)\\ &+ b(u, u - u_h, \Gamma_h v_h) - b(u - u_h, u - u_h, \Gamma_h v_h) = 0.\end{aligned}$$

Differentiating the above equations with respect to time t and noting that the definition of L^2-projection satisfies

$$(u_{tt} - P_h u_{tt}, \Gamma_h v_h) = 0,$$

we obtain that

$$\begin{aligned}&(u_{tt} - u_{htt}, \Gamma_h v_h) + \mathcal{C}_h((P_h u_t - u_h, p_t - J_h p_t); (v_h, q_h)) + b(u_t - u_{ht}, u, \Gamma_h v_h)\\ &\quad + b(u - u_h, u_t, \Gamma_h v_h) + b(u_t, u - u_h, \Gamma_h v_h) + b(u, u_t - u_{ht}, \Gamma_h v_h)\\ &\quad - b(u_t - u_{ht}, u - u_h, \Gamma_h v_h) - b(u - u_h, u_t - u_{ht}, \Gamma_h v_h)\\ &\quad = G(p_t, q_h) - A(u_t - P_h u_t, \Gamma_h v_h) - D(\Gamma_h v_h, p_t - J_h p_t)\\ &\quad\quad - d(u_t - P_h u_t, q_h) - G(p_t - J_h p_t, q_h).\end{aligned} \tag{5.26}$$

Taking $(v_h, q_h) = (e_t, 0) = (P_h u_t - u_{ht}, 0)$ in (5.26), we see that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\||e_t\||_0^2 + \nu\|e_t\|_1^2 + b(u_t - u_{ht}, u, \Gamma_h e_t - e_t) + b(u - u_h, u_t, \Gamma_h e_t - e_t)\\ &\quad + b(u_t, u - u_h, \Gamma_h e_t - e_t) + b(u, u_t - u_{ht}, \Gamma_h e_t - e_t)\\ &\quad - b(u_t - u_{ht}, u - u_h, \Gamma_h e_t - e_t) - b(u - u_h, u_t - u_{ht}, \Gamma_h e_t - e_t)\\ &\quad + b(u_t - u_{ht}, u, e_t) + b(u - u_h, u_t, e_t) + b(u_t, u - u_h, e_t)\\ &\quad + b(u, u_t - u_{ht}, e_t) - b(u_t - u_{ht}, u - u_h, e_t) - b(u - u_h, u_t - u_{ht}, e_t)\\ &\quad = -A(u_t - P_h u_t, \Gamma_h e_t) - D(\Gamma_h e_t, p_t - J_h p_t).\end{aligned} \tag{5.27}$$

By Lemma 4.6 and the Young inequality, it follows that

$$\begin{aligned}&|b(u_t - u_{ht}, u, \Gamma_h e_t - e_t) + b(u, u_t - u_{ht}, \Gamma_h e_t - e_t)|\\ &\quad \le c\|u_t - u_{ht}\|_1\|u\|_2\|\Gamma_h e_t - e_t\|_0\\ &\quad \le \frac{\nu}{16}\|e_t\|_1^2 + ch^2\left(\|u_t\|_1^2 + \|u_{ht}\|_1^2\right),\\ &|b(u - u_h, u_t, \Gamma_h e_t - e_t) + b(u_t, u - u_h, \Gamma_h e_t - e_t)|\\ &\quad \le c\|u - u_h\|_1\|u_t\|_2\|\Gamma_h e_t - e_t\|_0\\ &\quad \le \frac{\nu}{16}\|e_t\|_1^2 + ch^2\|u_t\|_2^2\left(\|u\|_1^2 + \|u_h\|_1^2\right).\end{aligned}$$

Applying (2.3), (3.3) and (4.7), we estimate as follows:

$$
\begin{aligned}
&|b(u_t - u_{ht}, u - u_h, \Gamma_h e_t - e_t)| \\
&\quad \le ch^{1/2}\left(\|u_{ht}\|_0^{1/2}\|u_{ht}\|_1^{1/2} + \|u_t\|_1\right)\|u - u_h\|_1\|\Gamma_h e_t - e_t\|_0 \\
&\quad \le c(\|u_{ht}\|_0 + \|u_t\|_1)\|u - u_h\|_1\|\Gamma_h e_t - e_t\|_0 \\
&\quad \le \frac{\nu}{16}\|e_t\|_1^2 + c\left(\|u_{ht}\|_0^2 + \|u_t\|_1^2\right)\|u - u_h\|_1^2.
\end{aligned}
$$

Thanks to (2.8) and (2.9), (2.3), and the Young inequality, we find that

$$
\begin{aligned}
&|b(u_t - u_{ht}, u, e_t) + b(u, u_t - u_{ht}, e_t)| \\
&\quad \le c\|u\|_2\|e_t\|_1(\|u_t - u_{ht}\|_0 + \|e_t\|_0) \\
&\quad \le \frac{\nu}{16}\|e_t\|_1^2 + c\|u\|_2^2\|e_t\|_0^2 + ch^2\left(\|u_t\|_2^2 + \|p_t\|_1^2\right), \\
&|b(u - u_h, u_t, e_t)| + |b(u_t, u - u_h, e_t)| \\
&\quad \le c\|u_t\|_1\|e_t\|_1\|u - u_h\|_1 \\
&\quad \le \frac{\nu}{16}\|e_t\|_1^2 + c\|u_t\|_1^2\|u - u_h\|_1^2.
\end{aligned}
$$

Also, the same approach described above can be used to obtain the following estimate:

$$
\begin{aligned}
&|b(u_t - u_{ht}, u - u_h, e_t)| + |b(u - u_h, u_t - u_{ht}, e_t)| \\
&\quad = |b(E_t, u - u_h, e_t) + b(u - u_h, E_t, e_t) + b(e_t, u - u_h, e_t)| \\
&\quad \le c\|E_t\|_1\|u - u_h\|_1\|e_t\|_1 \\
&\qquad + c\left(\|e_t\|_0\|e_t\|_1\|u - u_h\|_1 + \|e_t\|_0^{1/2}\|e_t\|_1^{3/2}\|u - u_h\|_0^{1/2}\|u - u_h\|_1^{1/2}\right) \\
&\quad \le \frac{\nu}{16}\|e_t\|_1^2 + c(\|u_t\|_1^2 + \|R_h(u, p)\|_1^2)\|u - u_h\|_1^2 \\
&\qquad + c\|e_t\|_0^2\|u - u_h\|_1^2\left(1 + \|u - u_h\|_0^2\right) \\
&\quad \le \frac{\nu}{16}\|e_t\|_1^2 + c\left(\|u_t\|_1^2 + \|u\|_2^2 + \|p\|_1^2\right)\|u - u_h\|_1^2 + c\|e_t\|_0^2\left(\|u\|_1^2 + \|u_h\|_1^2\right).
\end{aligned}
$$

Similarly, with the same approach as for Lemma 4.5, we arrive at

$$
\begin{aligned}
|A(u_t - P_h u_t, \Gamma_h e_t)| &\le ch\|u_t\|_2\|e_t\|_1 \le \frac{\nu}{16}\|e_t\|_1^2 + ch^2\|u_t\|_2^2, \\
|D(\Gamma_h e_t, p_t - J_h p_t)| &\le ch\|p_t\|_1\|e_t\|_1 \le \frac{\nu}{16}\|e_t\|_1^2 + ch^2\|p_t\|_2^2.
\end{aligned}
$$

Also, using the inverse inequality (3.3), Lemmas 4.1, 4.6, (5.3), and the Young inequality leads to

$$
\begin{aligned}
&|b(u-u_h, u_t-u_{ht}, \Gamma_h e_t - e_t)| \\
&\quad = |b(u-u_h, u_t, \Gamma_h e_t - e_t) - b(u-u_h, u_{ht}, \Gamma_h e_t - e_t)| \\
&\quad \le c\|u_t\|_2\|u-u_h\|_1\|\Gamma_h e_t - e_t\|_0 + ch^{1/2}\|u_{ht}\|_0^{1/2}\|u_{ht}\|_1^{1/2}\|u-u_h\|_1\|e_t\|_1 \\
&\quad \le ch(\|u_t\|_2 + \|u_{ht}\|_0)\|e_t\|_1\|u-u_h\|_1 \\
&\quad \le \frac{\nu}{16}\|e_t\|_1^2 + ch^2\left(\|u_t\|_2^2 + \|u_{ht}\|_0^2\right).
\end{aligned}
$$

Applying these inequalities, Lemmas 2.2 and 5.1–5.3, and the triangle inequality yields

$$
\begin{aligned}
\frac{d}{dt}\||e_t\||_0^2 + \nu\|e_t\|_1^2 \le c\Big\{& h^2\left(\|u_t\|_2^2 + \|p_t\|_1^2 + \|u_{ht}\|_1^2\right) + \left(\|u_h\|_1^2 + \|u\|_2^2\right)\|e_t\|_0^2 \\
&+ \left(\|u_t\|_2^2 + \|p_t\|_1^2 + \|u_{ht}\|_0^2\right)\|u-u_h\|_1^2\Big\}. \qquad (5.28)
\end{aligned}
$$

Substituting these inequalities into (5.27) by $\tau(t)$, integrating from 0 to t, we conclude that

$$
\begin{aligned}
&\tau(t)\|e_t(t)\|_0^2 + \nu\int_0^t \tau(s)\|e_t\|_1^2 ds \\
&\qquad \le c\Big\{h^2\int_0^t \tau(s)\left(\|u_t\|_2^2 + \|p_t\|_1^2 + \|u_{ht}\|_1^2\right)ds \\
&\qquad\quad + \|u(t)-u_h(t)\|_1^2\int_0^t \tau(s)\left(\|u_t\|_2^2 + \|u_{ht}\|_0^2 + \|p_t\|_1^2\right)ds \\
&\qquad\quad + \int_0^t \tau(s)\left(\|u_h\|_1^2 + \|u\|_2^2\right)\|e_t\|_0^2 ds + \int_0^t \|e_t\|_0^2 ds\Big\},
\end{aligned}
$$

which, together with (3.1), (3.2), (5.24), Lemmas 2.1 and 2.2 and 5.1–5.3, completes the proof. □

Lemma 5.5 *Under the assumptions of* (A1)–(A2)*, it holds that, for* $t \in [0, T]$*,*

$$\tau^{1/2}(t)\|p(t) - p_h(t)\|_0 \le ch. \qquad (5.29)$$

Proof Setting $(e(t), \eta(t)) = (R_h(u(t), p(t)) - u_h(t), Q_h(u(t), p(t)) - p_h(t))$ and using (2.3), (3.3), (4.21), (4.24) and (5.7) yield that

$$\begin{aligned}
&\beta\|\eta(t)\|_0 \\
&\quad\le \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{\mathcal{C}_h((e,\eta);(v_h,q_h))}{\|v_h\|_1+\|q_h\|_0} \\
&\quad= \sup_{(v_h,q_h)\in(X_h,M_h)} \left\{ \frac{(u_t-u_{ht},\Gamma_h v_h)+b(u-u_h,u,v_h)+b(u-u_h,u,\Gamma_h v_h - v_h)}{\|v_h\|_1+\|q_h\|_0} \right. \\
&\qquad\qquad \left. + \frac{(b(u_h,u-u_h,\Gamma_h v_h - v_h)+b(u_h,u-u_h,v_h)}{\|v_h\|_1+\|q_h\|_0} \right\}. \qquad (5.30)
\end{aligned}$$

In view of (2.8) and (2.9), Lemma 4.6, it follows that

$$\begin{aligned}
|b(u-u_h,u,v_h)+b(u_h,u-u_h,v_h)| &\le c(\|u\|_1+\|u_h\|_1)\|u-u_h\|_1\|v_h\|_1, \\
|b(u-u_h,u,\Gamma_h v_h - v_h)| &\le c\|u\|_2\|u-u_h\|_1\|\Gamma_h v_h - v_h\|_1 \\
&\le ch\|u\|_2\|u-u_h\|_1\|v_h\|_1.
\end{aligned}$$

The combination of Lemmas 5.3 and 5.4 shows that

$$\tau^{1/2}(t)\|\eta(t)\|_0 \le c\tau^{1/2}(t)(\|u_t(t)-u_{ht}(t)\|_0+\|u(t)-u_h(t)\|_1) \le ch. \qquad (5.31)$$

As a result, applying Lemmas 2.2 and 4.5 and (5.31) gives

$$\tau^{1/2}(t)\|p(t)-p_h(t)\|_0 \le \tau^{1/2}(t)(\|\eta(t)\|_0 + c\|p(t)-Q_h(u(t),p(t))\|_0) \le ch,$$

which is (5.29). □

Theorem 5.6 *Under the assumptions of* (5.1) *and* (A1)–(A2)*, it holds that, for* $t\in[0,T]$,

$$\|u(t)-u_h(t)\|_1 + \tau^{1/2}(t)\|p(t)-p_h(t)\|_0 \le ch. \qquad (5.32)$$

This theorem follows from Lemmas 5.3 and 5.5.

6 L^2-error estimates

Observed from the previous analysis in Section 5, we can find that different analysis techniques are applied to the finite volume method from those for the finite element method for the transient Navier–Stokes equations. As for the L^2-norm estimate for velocity, we must take special care of the optimal order convergence analysis because there is only an $O(h)$ error between the test functions of the finite element method and those of the finite volume method.

In this section we estimate the error $\|u-u_h\|_0$ using a parabolic duality argument for a backward-in-time linearized Navier–Stokes problem [21, 22, 24].

The dual problem is to seek $(\Phi(t), \Psi(t)) \in X \times M$ such that, for $t \in [0, T]$ and $g \in L^2(0, T; Y)$,

$$(v, \Phi_t) - B\big((v, q); (\Phi, \Psi)\big) - b(u, v, \Phi) - b(v, u, \Phi) = (v, u - u_h), \tag{6.1}$$

for all $(v, q) \in (X, M)$, with $\Phi(T) = 0$. This problem is well-posed and has a unique solution (Φ, Ψ) satisfying [24]

$$\Phi \in C(0, T; V) \cap L^2(0, T; D(A)) \cap H^1(0, T; Y), \;\; \Psi \in L^2(0, T; H^1(\Omega) \cap M).$$

We recall the following regularity result [24].

Lemma 6.1 *Under the assumption of* (A1)*, the solution* (Φ, Ψ) *of* (6.1) *satisfies*

$$\sup_{0\le t\le T} \|\Phi(t)\|_1^2 + \int_0^T \big(\|\Phi\|_2^2 + \|\Psi\|_1^2 + \|\Phi_t\|_0^2\big)\, dt \le c\int_0^T \|u - u_h\|_0^2\, dt. \tag{6.2}$$

Based on the results provided in Section 5, a duality argument is applied to overcome the lower order convergence rate of the Stokes projection defined in (4.22) by involving the Stokes projection defined in (3.13). Then, optimal analysis is provided in the following two Lemmas.

Lemma 6.2 *Under the assumptions of* (A1)–(A2)*, it holds that, for* $t \in [0, T]$*,*

$$\int_0^T \|u - u_h\|_0^2 ds \le ch^4. \tag{6.3}$$

Proof Let $(\Phi_h(t), \Psi_h(t))$ be the dual Galerkin projection in (X_h, M_h) of $(\Phi(t), \Psi(t))$ such that

$$\mathcal{B}_h((v_h, q_h)); (\Phi_h, \Psi_h)) = B((v_h, q_h); (\Phi, \Psi)) \quad \forall\, (v_h, q_h) \in (X_h, M_h), \tag{6.4}$$

with

$$\begin{aligned} &\|\Phi_h\|_1 + \|\Psi_h\|_0 \le c(\|\Phi\|_1 + \|\Psi\|_0),\\ &\|\Phi - \Phi_h\|_0 + h(\|\Phi - \Phi_h\|_1 + \|\Psi - \Psi_h\|_0) \le ch^2(\|\Phi\|_2 + \|\Psi\|_1). \end{aligned} \tag{6.5}$$

Taking $(v_h, q_h) = (\Phi_h, \Psi_h)$ in (5.7), we can find that

$$\begin{aligned} (e_t, \Gamma_h\Phi_h) + \mathcal{C}_h\big((e, \eta); (\Phi_h, \Psi_h)\big) + b(u, e, \Gamma_h\Phi_h)&\\ + b(e, u, \Gamma_h\Phi_h) - b(e, e, \Gamma_h\Phi_h) = 0&, \end{aligned} \tag{6.6}$$

where $(e, \eta) = (u - u_h, p - p_h)$. Adding (6.6) and (6.1) with $(v, q) = (e, \eta)$, we see that

$$\begin{aligned} \|e\|_0^2 = &\frac{d}{dt}(e, \Phi) - (e_t, \Phi - \Gamma_h\Phi_h) + A(e, \Gamma_h\Phi_h) - a(e, \Phi_h) + D(\Gamma_h\Phi_h, \eta)\\ &+ d(\Phi_h, \eta) - a(e, \Phi - \Phi_h) - d(e, \Psi - \Psi_h) + d(\Phi - \Phi_h, \eta) + G(\eta, \Psi_h)\\ &- b(e, u, \Phi - \Gamma_h\Phi_h) - b(u, e, \Phi - \Gamma_h\Phi_h) - b(e, e, \Gamma_h\Phi_h). \end{aligned}$$

That is,

$$\begin{aligned}\|e\|_0^2 = \frac{d}{dt}(e, \Phi) - (e_t, \Phi - \Gamma_h\Phi_h) + A(u, \Gamma_h\Phi_h) - a(u, \Phi_h) + D(\Gamma_h\Phi_h, p) \\ + d(\Phi_h, p) - \mathcal{B}_h((e, \eta); (\Phi - \Phi_h, \Psi - \Psi_h)) + G(\eta, \Psi) \\ - b(e, u, \Phi - \Gamma_h\Phi_h) - b(u, e, \Phi - \Gamma_h\Phi_h) - b(e, e, \Gamma_h\Phi_h). \qquad (6.7)\end{aligned}$$

Noting that

$$\begin{aligned}|(e_t, \Phi - \Phi_h)| &\le ch^2(\|u_t\|_0 + \|u_{ht}\|_0)\|\Phi\|_2, \\ |(e_t, \Phi_h - \Gamma_h\Phi_h)| &= |(e_t - \Pi_h e_t, \Phi_h - \Gamma_h\Phi_h)| \\ &\le ch^2(\|u_t\|_1 + \|u_{ht}\|_1)\|\Phi\|_1,\end{aligned}$$

we have

$$\begin{aligned}|(e_t, \Phi - \Gamma_h\Phi_h)| &\le |(e_t, \Phi - \Phi_h)| + |(e_t, \Phi_h - \Gamma_h\Phi_h)| \\ &\le ch^2(\|u_t\|_1 + \|u_{ht}\|_1)\|\Phi\|_2.\end{aligned}$$

Thanks to (2.8) and (2.9), (4.32), and (6.5), we have

$$\begin{aligned}|b(e, u, \Phi - \Gamma_h\Phi_h)| &\le c\|u\|_2\|e\|_1(\|\Phi - \Phi_h\|_0 + \|\Phi_h - \Gamma_h\Phi_h\|_0) \\ &\le ch\|u\|_2\|e\|_1\|\Phi\|_1.\end{aligned}$$

Similarly,

$$\begin{aligned}|b(u, e, \Phi - \Gamma_h\Phi_h)| &\le |b(u, e, \Phi - \Phi_h)| + |b(u, e, \Phi_h - \Gamma_h\Phi_h)| \\ &\le c\|u\|_2\|e\|_1(\|\Phi - \Phi_h\|_0 + \|\Phi_h - \Gamma_h\Phi_h\|_0) \\ &\le ch\|u\|_2\|e\|_1\|\Phi\|_1.\end{aligned}$$

By a triangle inequality, (2.8) and (2.9), (3.3), (4.7), (4.31), and (6.5), it follows that

$$\begin{aligned}|b(e, e, \Gamma_h\Phi_h)| &\le |b(e, e, \Phi_h) + b(e, u, \Phi_h - \Gamma_h\Phi_h) + b(e, u_h, \Phi_h - \Gamma_h\Phi_h)| \\ &\le c\|e\|_1^2\|\Phi\|_1 + ch\|u\|_2\|e\|_1\|\Phi\|_1 + ch^{3/2}\|u_h\|_0^{1/2}\|u_h\|_1^{1/2}\|e\|_1\|\Phi\|_1 \\ &\le c\|e\|_1^2\|\Phi\|_1 + ch(\|u\|_2 + \|u_h\|_0)\|e\|_1\|\Phi\|_1.\end{aligned}$$

Also, the same approach as (5.9) shows that

$$\begin{aligned}&A(u, \Gamma_h\Phi_h) - a(u, \Phi_h) + D(\Gamma_h\Psi_h, p) + d(\Phi_h, p) \\ &= (f - (u \cdot \nabla)u - u_t, \Gamma_h\Phi_h - \Phi_h) \\ &\le ch^2\left(\|f\|_1 + \|u_t\|_1 + \|u\|_0^{1/2}\|u\|_2^{3/2}\right)\|\Phi\|_1.\end{aligned}$$

Then, using (3.3), (6.4), and (6.5), gives

$$\begin{aligned}|\mathcal{B}_h((e,\eta);(\Phi-\Phi_h,\Psi-\Psi_h))-G(\eta,\Psi)|\\ &= \mathcal{B}_h((u-\bar{R}_h(u,p),p-\bar{Q}_h(u,p));(\Phi-\Phi_h,\Psi-\Psi_h))+G(\eta,\Psi)\\ &\le ch^2(\|u\|_2+\|p\|_1)(\|\Phi\|_2+\|\Psi\|_1)+ch\|\eta\|_0\|\Psi\|_1.\end{aligned}$$

The combination of these estimates with (A2), (6.4), (6.5), and (6.7), we see that

$$\begin{aligned}\|e\|_0^2 \le \frac{d}{dt}(e,\Phi)+c\Big\{&h^2\left(\|u_t\|_1+\|u_{ht}\|_1+\|f\|_1+\|u\|_0^{1/2}\|u\|_2^{3/2}\right)\|\Phi\|_2\\ &+\|e\|_1^2\|\Phi\|_1+h(\|u\|_2+\|u_h\|_0)\|e\|_1\|\Phi\|_1\\ &+h^2(\|u\|_2+\|p\|_1)(\|\Phi\|_2+\|\Psi\|_1)+h\|\eta\|_0\|\Psi\|_1\Big\}\\ \le \frac{d}{dt}(e,\Phi)+c\Big\{&h^2\left(\|u_t\|_1+\|u_{ht}\|_1+\|f\|_1+\|u\|_0^{1/2}\|u\|_2^{3/2}\right)(\|\Phi\|_2+\|\Psi\|_1)\\ &+\|e\|_1^2\|\Phi\|_1+h(\|\Phi\|_2+\|\Psi\|_1)(\|e\|_1+\|\eta\|_0)\Big\}.\end{aligned} \tag{6.8}$$

Integrating (6.8) from 0 to T yields

$$\begin{aligned}&\int_0^T\|e(s)\|_0^2\,ds\\ &\le c\Big\{h^2\left(\int_0^T\left(\|u_t\|_1^2+\|u_{ht}\|_1^2+\|f\|_1+\|u\|_0^{1/2}\|u\|_2^{3/2}\right)ds\right)^{1/2}\\ &\quad\times\left(\int_0^T(\|\Phi\|_2^2+\|\Psi\|_1^2)\,ds\right)^{1/2}+h\left(\int_0^T(\|e\|_1^2+\|\eta\|_0^2)\,ds\right)^{1/2}\\ &\quad\times\left(\int_0^T(\|\Phi\|_2^2+\|\Psi\|_1^2)\,ds\right)^{1/2}-(e(0),\Phi(0))\\ &\quad+\sup_{0\le t\le T}\|\Phi(t)\|_1\int_0^T\|e\|_1^2\,ds\Big\}.\end{aligned} \tag{6.9}$$

In addition, by the definition of the projection R_h and the initial approximation, we have

$$|(e(0),\Phi(0))|=|(u_0-P_hu_0,\Phi(0))|\le ch^2(\|u_0\|_2+\|p_0\|_1)\|\Phi(0)\|_1. \tag{6.10}$$

Combining (A2), (5.3) and (5.4), (5.12), (6.2), (6.9), and (6.10) completes the proof of (6.3). □

Lemma 6.3 *Under the assumptions of* (A1)–(A2)*, it holds that, for* $t\in[0,T]$,

$$\|u(t)-u_h(t)\|_0\le c\tau^{-1/2}(t)h^2. \tag{6.11}$$

Proof Taking $(v_h, q_h) = (e, 0) = (\bar{R}_h(u, p) - u_h, 0)$ and setting $E = u - \bar{R}_h(u, p)$ in (5.7), we see that

$$\frac{1}{2}\frac{d}{dt}|||e|||_0^2 + \nu\|e\|_1^2 + 2G(\eta,\eta) + b(u, u-u_h, \Gamma_h e - e) + b(u-u_h, u, \Gamma_h e - e) \\ - b(u-u_h, u-u_h, \Gamma_h e - e) + b(u, u-u_h, e) + b(u-u_h, u, e) \\ - b(u-u_h, u-u_h, e) = -(E_t, \Gamma_h e) - A(E, \Gamma_h e) - D(\Gamma_h e, p - \bar{Q}_h(u, p)). \tag{6.12}$$

Clearly, using Lemmas 3.2 and the Young inequality, gives

$$|(E_t, \Gamma_h e)| \le \|E_t\|_0 \|e\|_0 \le ch^2(\|u_t\|_2 + \|p_t\|_1)\|e\|_0 \\ \le ch^4\left(\|u_t\|_2^2 + \|p_t\|_1^2\right) + \|e\|_0^2.$$

Thanks to (2.8) and (2.9), we have

$$|b(u, u-u_h, e) + b(u-u_h, u, e)| \le c\|u\|_2\|e\|_1\|u-u_h\|_0 \\ \le \frac{\nu}{10}\|e\|_1^2 + c\|u\|_2^2\|u-u_h\|_0^2, \\ |b(u-u_h, u-u_h, e)| \le \|u-u_h\|_1^2\|e\|_1 \\ \le \frac{\nu}{10}\|e\|_1^2 + c\|u-u_h\|_1^4.$$

By (4.32), it follows that

$$|b(u, u-u_h, \Gamma_h e - e) + b(u-u_h, u, \Gamma_h e - e)| \le c\|u\|_2\|u-u_h\|_1\|\Gamma_h e - e\|_0 \\ \le ch^2\|u\|_2\|u-u_h\|_1^2 + \frac{\nu}{10}\|e\|_1^2.$$

Obviously, we deduce from L^∞ estimate in (2.3), the inverse inequality (3.3) and Lemma 4.1 that

$$|b(u-u_h, u-u_h, \Gamma_h e - e)| \le ch\|u-u_h\|_0(\|u\|_{L^\infty} + \|u_h\|_{L^\infty})\|\Gamma_h e - e\|_0 \\ \le ch^{1/2}\|u-u_h\|_0(\|u\|_2 + \|u_h\|_1)\|e\|_1 \\ \le ch\|u-u_h\|_0^2\left(\|u\|_2^2 + \|u_h\|_1^2\right) + \frac{\nu}{10}\|e\|_1^2.$$

Also, it follows from the same approach as (5.9) and the Young inequality that

$$|A(E, \Gamma_h e) + D(\Gamma_h e, p - \bar{Q}_h(u, p))| \\ = |([f - \Pi_h f] - [(u\cdot\nabla)u - \Pi_h(u\cdot\nabla)u] - [u_t - \Pi_h u_t], \Gamma_h e - e)| \\ \le ch^2\left(\|f\|_1 + \|u\|_0^{1/2}\|u\|_2^{3/2} + \|u_t\|_1\right)\|e\|_1 \\ \le \frac{\nu}{10}\|e\|_1^2 + ch^4\left(\|f\|_1^2 + \|u\|_0\|u\|_2^3 + \|u_t\|_1^2\right).$$

Therefore, combining these estimates with (6.12) gives

$$\frac{d}{dt}\||e\||_0^2+\nu\|e\|_1^2+G(\eta,\eta)$$
$$\leq c\Big\{h^4\left(\|u_t\|_2^2+\|p_t\|_1^2+\|f\|_1^2+\|u\|_0\|u\|_2^3\right)+\|e\|_0^2+\|u\|_2^2\|u-u_h\|_0^2$$
$$+\|u-u_h\|_1^4+h^2\|u-u_h\|_1^2+\|u-u_h\|_0\|u-u_h\|_1^3\Big\}. \tag{6.13}$$

Note that

$$\int_0^T\|e\|_0^2\,ds\leq\int_0^T\|u-u_h\|_0^2\,ds+\int_0^T\|E\|_0^2\,ds$$
$$\leq\int_0^T\|u-u_h\|_0^2\,ds+ch^4\int_0^T\left(\|u\|_2^2+\|p\|_1^2\right)\,ds\leq ch^4. \tag{6.14}$$

Multiplying (6.13) by $\tau(t)$, integrating from 0 to t, and using (6.13), (6.14), and Lemmas 2.2 and 4.2, we obtain

$$\tau(t)\|e(t)\|_0^2+\int_0^t\tau(s)\left(\nu\|e\|_1^2+G(\eta,\eta)\right)ds$$
$$\leq c\Big\{\int_0^t\|e\|_0^2ds+h^4\int_0^t\tau(s)\left(\|u_t\|_2^2+\|p_t\|_1^2\right)ds+\|u\|_2^2\int_0^t\|u-u_h\|_0^2ds$$
$$+\int_0^t\|u-u_h\|_1^4ds+h^2\int_0^t\|u-u_h\|_1^2ds+\int_0^t\|u-u_h\|_0\|u-u_h\|_1^3ds\Big\}$$
$$\leq ch^4. \tag{6.15}$$

Using Lemmas 2.2 and 3.2, we have

$$\|u(t)-\bar{R}_h(u(t),p(t))\|_0^2\leq ch^4\left(\|u(t)\|_2^2+\|p(t)\|_1^2\right)\leq ch^4, \tag{6.16}$$

which, together with (6.15), yields (6.11). □

The next theorem follows from Lemmas 5.6 and 6.3.

Theorem 6.4 *Under the assumptions of* (A1)–(A2), *it holds that, for* $t\in[0,T]$,

$$\tau^{1/2}(t)\|u(t)-u_h(t)\|_0+h(\|u(t)-u_h(t)\|_1+\tau^{1/2}(t)\|p(t)-p_h(t)\|_0)\leq ch^2. \tag{6.17}$$

7 Numerical experiments

Two examples are presented to check the stability and convergence properties of the stabilized finite volume method for the transient Navier–Stokes

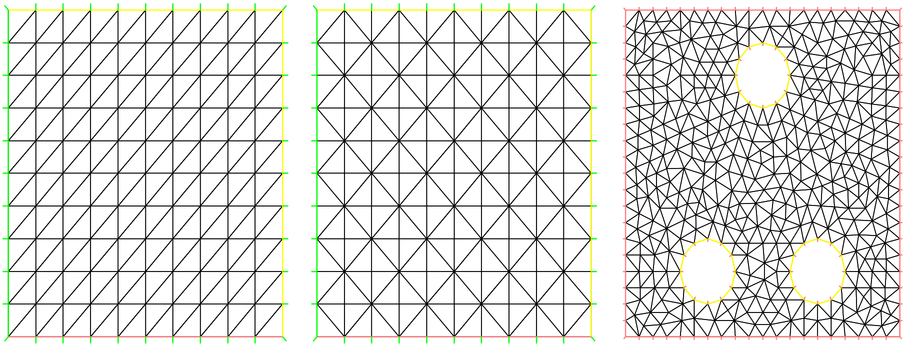


Fig. 3 Uniform and unstructured triangulations of Ω into triangles

equations over different types of meshes. Here the most robust backward Euler scheme is applied to solve the transient Navier–Stokes equations in order to test the efficiency of the present stabilized finite volume method: The time variable is discretized by the common time difference scheme, and each of these problems is then solved by the spatial discretization method as described in Section 5.

Example I The exact solution is designed to test the finite volume method on two kinds of uniform grids: a box grid and a criss-cross grid (see the first and second grids in Fig. 3). We consider the transient Navier–Stokes equations on $\Omega = (0, 1) \times (0, 1)$, with the fluid viscosity $\nu = 0.01$ and the body force $f(x, t)$ such that the true solution is

$$u(x, t) = (u_1(x_1, x_2, t), u_2(x_1, x_2, t)), \quad p(x, t) = 10(2x_1 - 1)(2x_2 - 1)\cos(t),$$

$$u_1(x_1, x_2, t) = 10x_1^2(x_1 - 1)^2 x_2(x_2 - 1)(2x_2 - 1)\cos(t),$$

$$u_2(x_1, x_2, t) = -10x_1(x_1 - 1)(2x_1 - 1)x_2^2(x_2 - 1)^2\cos(t).$$

Moreover, the initial value $u_0 = (u_1(x, 0), u_2(x, 0))$ is set by the value of the above exact solution $u_1(x, t)$ and $u_2(x, t)$ at $t = 0$ satisfying assumption (A2) on the exact solution. The errors in the L^2- and H^1-norms for velocity and in the

Table 1 The results for the finite volume method ($\nu = 0.01$ on uniform mesh-1)

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	e_h
10	0.354496	0.891072	0.0301807	0.00276376
20	0.0829056	0.333391	0.00762608	0.000386261
30	0.0356544	0.191583	0.00348278	0.000115246
40	0.0196911	0.130815	0.00201759	4.83689e-005
50	0.012454	0.0975029	0.00133284	2.45913e-005
60	0.00857769	0.0767381	0.000956569	1.41373e-005
70	0.00626485	0.0627375	0.000726293	8.85208e-006
80	0.004776	0.0527669	0.00057414	5.90172e-006
90	0.00376193	0.0453648	0.000467739	4.12822e-006

Table 2 The results for the finite volume method ($\nu = 0.01$ on uniform mesh-2)

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	e_h
10	0.270911	0.692647	0.0620917	0.00311491
20	0.0667608	0.285343	0.0135366	0.000488715
30	0.0291311	0.168417	0.00577026	0.000153989
40	0.0161934	0.116187	0.0032025	6.6697e-005
50	0.0102811	0.0872674	0.00204596	3.46346e-005
60	0.0070996	0.0691545	0.0014264	2.02176e-005
70	0.00519519	0.0568705	0.00105532	1.2806e-005
80	0.00396631	0.0480595	0.000815067	8.6147e-006
90	0.00312776	0.0414693	0.00065032	6.06916e-006

L^2-norm for pressure are shown in Tables 1 and 2 and Fig. 4. The results show that optimal order error estimates are obtained for the transient Navier–Stokes equations approximated by the present finite volume method. In addition, superconvergence results occur for the pressure in the L^2-norm for both the

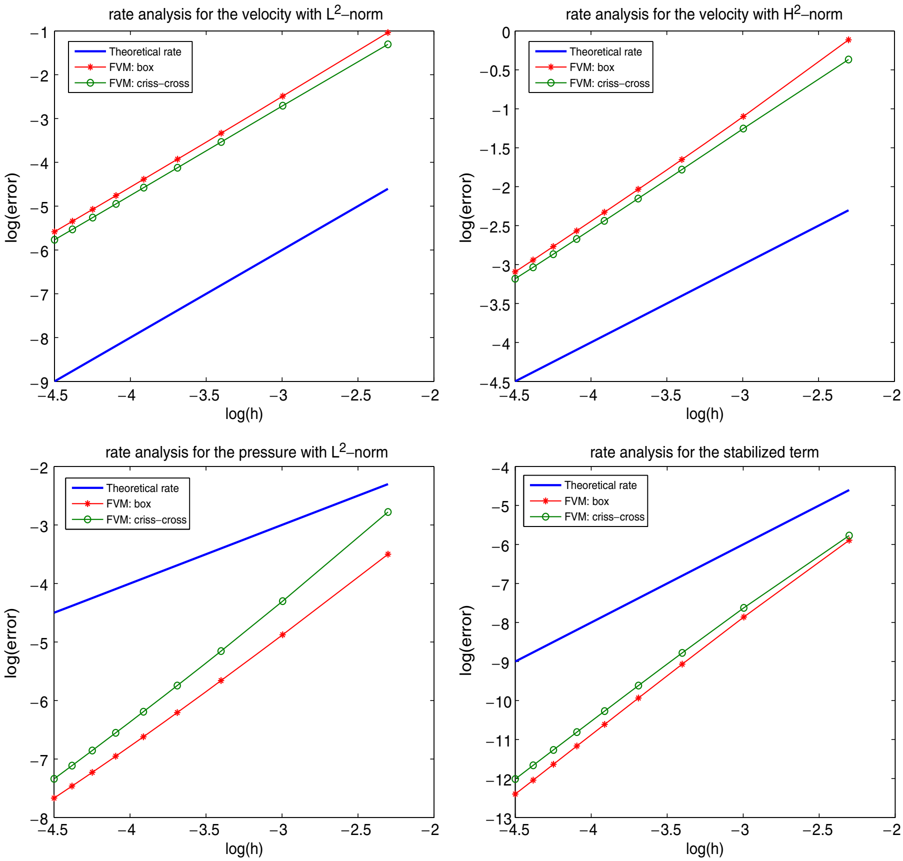


Fig. 4 Comparison of rates for the finite volume method on the *box* and *criss-cross grids*

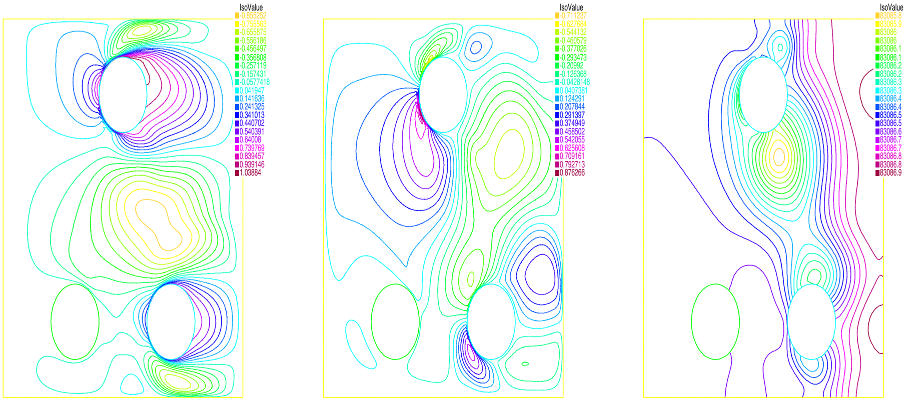


Fig. 5 Streamline pattern: finite element method, Taylor–Hood element

box grid and the criss-cross grid. Also, the stabilized term is computed, element by element, by using

$$e_h = \max_{K \in K_h} \left| \int_{\partial K} u_h \cdot n \, ds \right| = \max_{K \in K_h} \left| \int_K (\mathrm{div} u_h - \mathrm{div} u) ds \right| .$$

From Tables 1 and 2 and Fig. 4, there is superconvergence result on the stabilized term. Along with the mesh scale h decrease, the error of the stabilized term approaches zero. Thus, it seems that there is no negative effect on the original model.

Example II Cavity flows have widely been used as test cases for validating an incompressible fluid dynamics algorithm. It is well known that corner singularities for two-dimensional fluid flows are very important because most examples of physical interest have corners. In this example, we consider

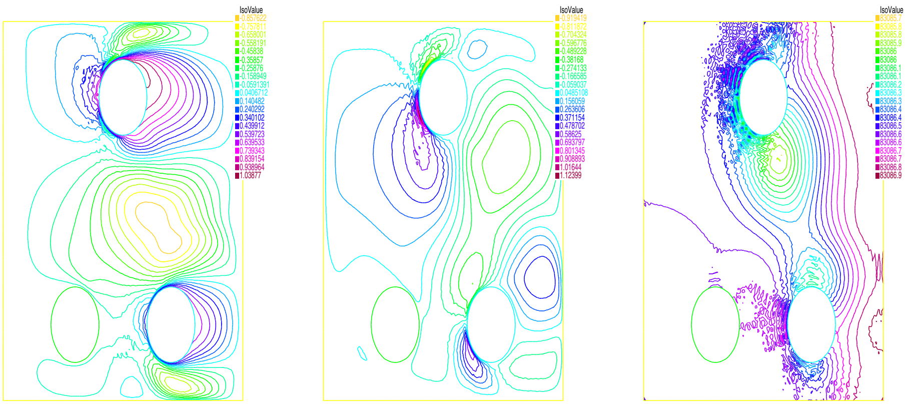


Fig. 6 Streamline pattern: stabilized finite element method, $P_1 - P_1$

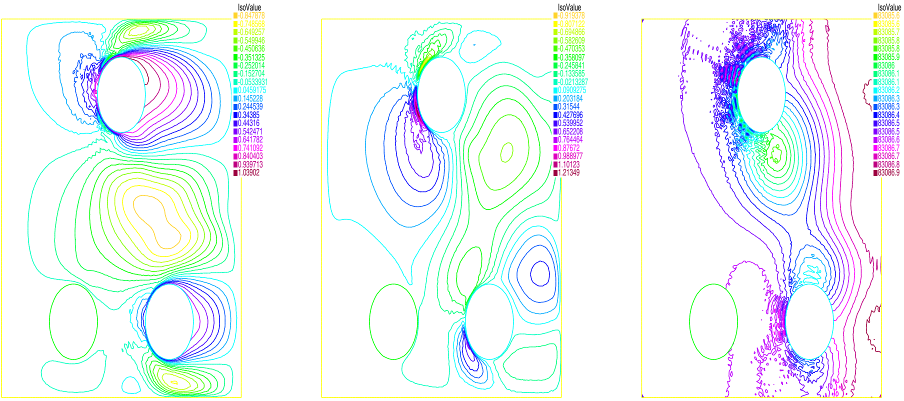


Fig. 7 Streamline pattern: stabilized finite volume method, $P_1 - P_1$

the driven flow in a rectangular cavity when the top surface moves with a enough smooth velocity with $u_0 = (x_1^2 + x_2^2, 0)$ satisfying (A2) along its length. Also, an unstructured grid is applied with three holes in the domain. Physical phenomenon is more complicated around the holes.

In this case, we compute an approximate solution for $\nu = 0.001$ on the unstructured grid. The methods tested include the finite element method with the $P_2 - P_1$ pair (the Taylor–Hood element), the stabilized finite element method with the $P_1 - P_1$ pair (see the third section), and the present stabilized finite volume method. The stable Taylor–Hood element performs the best since it employs the higher order finite element pair for the velocity and pressure. This finite element pair also has a superconvergence performance for both velocity and pressure. Since we do not have an exact solution in this case, we prefer to rank the accuracies of the finite element method approximated by Taylor–Hood element as the "exact" solution. Figures 5, 6 and 7 with 10,304° of freedom shows that the results of two velocity components and pressure of the present finite volume method is completely in good agreement with those of finite element method with Taylor–Hood element and stabilized $P_1 - P_1$ element.

In conclusion, numerical results reported completely agree with the theoretical results established in this paper.

Acknowledgements We thank the anonymous referees very much for their helpful comments and suggestions, which lead to substantial improvements of the presentation.

References

1. Bank, R.E., Rose, D.J.: Some error estimates for the box method. SIAM J. Numer. Anal. **24**, 777–787 (1987)
2. Bochev, P., Dohrmann, C.R., Gunzburger, M.D.: A computational study of stabilized, low 581 order C^0 finite element approximations of Darcy equations. Comput. Mech. **38**, 323–333 (2006)

3. Bochev, P., Dohrmann, C.R., Gunzburger, M.D.: Stabilization of low-order mixed finite elements for the Stokes equations. SIAM J. Numer. Anal. **44**, 82–101 (2006)
4. Cai, Z.: On the finite volume method. Numer. Math. **58**, 713–735 (1991)
5. Chen, Z.: The control volume finite element methods and their applications to multiphase flow. Netw. Heterog. Media **1**, 689–706 (2006)
6. Chen, Z., Li, R., Zhou, A.: A note on the optimal L^2-estimate of finite volume element method. Adv. Comput. Math. **16**, 291–303 (2002)
7. Chou, S.H., Li, Q.: Error estimates in L^2, H^1 and L^∞ in co-volume methods for elliptic and parabolic problems: a unified approach. Math. Comput. **69**, 103–120 (2000)
8. Chatzipantelidis, P.: A finite volume method based on the Crouzeix–Raviart element for elliptic PDEs in two dimensions. Numer. Math. **82**, 409–432 (1999)
9. Chatzipantelidis, P., Lazarov, R.D., Thomée, V.: Error estimates for a finite volume method for parabolic equations in convex polygonal domains. Numer. Methods Partial Differ. Equ. **20**, 650–674 (2004)
10. Chen, Z.: Finite Element Methods and Their Applications. Spring-Verlag, Heidelberg (2005)
11. Chou, S.H.: Analysis and convergence of a covolume method for the generalized Stokes problem. Math. Comput. **66**, 85–104 (1997)
12. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
13. Dohrmann, C.R., Bochev, P.: A stabilized finite element method for the Stokes problem based on polynomial pressure projections. Int. J. Numer. Methods Fluids **46**, 183–201 (2004)
14. Ewing, R.E., Lazarov, R.D., Lin, Y.: Finite volume element approximations of nonlocal reactive flows in porous media. Numer. Methods Partial Differ. Equ. **16**, 285–311 (2000)
15. Ewing, R.E., Lin, T., Lin, Y.: On the accuracy of the finite volume element method based on piecewise linear polynomials. SIAM J. Numer. Anal. **39**, 1865–1888 (2002)
16. Eymard, R., Gallouët, T., Herbin, R.: Finite volume methods. Handbook of Numerical Analysis VII, vol. 46, pp. 713–1020. North Holland (2000)
17. Eymard, R., Herbin, R., Latché, J.C.: On a stabilized colocated finite volume scheme for the Stokes problem. ESAIM: M2AN **40**, 501–528 (2006)
18. Eymard, R., Herbin, R., Latché, J.C., Piar, B.: Convergence analysis of a locally stabilized collocated finite volume scheme for incompressible flows. ESAIM: M2AN **43**, 889–927 (2009)
19. Girault, V., Raviart, P.A.: Finite Element Methods for Navier–Stokes Equations: Theory and Algorithms. Springer-Verlag, Berlin (1986)
20. He, Y., Li, J.: A stabilized finite element method based on local polynomial pressure projection for the stationary Navier–Stokes equation. Appl. Numer. Math. **58**, 1503–1514 (2008)
21. He, Y., Lin, Y., Sun, W.: Stabilized finite element method for the non-stationary Navier–Stokes problem. Discrete Continuous Dyn. Syst. Ser. B **6**, 41–68 (2006)
22. Heywood, J.G., Rannacher, R.: Finite element approximation of the nonstationary Navier–Stokes problem I: Regularity of solutions and second-order error estimates for spatial discretization. SIAM J. Numer. Anal. **19**, 275–311 (1982)
23. Huang, J., Xi, S.: On the finite volume element method for general self-adjoint elliptic problems. SIAM J. Numer. Anal. **35**, 1762–1774 (1998)
24. Hill, A.T., Süli, E.: Approximation of the global attractor for the incompressible Navier–Stokes problem. IMA J. Numer. Anal. **20**, 633–667 (2000)
25. Li, J., Chen, Z.: A new stabilized finite volume method for the stationary Stokes equations. Adv. Comput. Math. **30**, 141–152 (2009)
26. Li, J., He, Y.: A new stabilized finite element method based on two local Gauss integration for the Stokes equations. J. Comput. Appl. Math. **214**, 58–65 (2008)
27. Li, J., He, Y., Chen, Z.: A new stabilized finite element method for the transient Navier–Stokes equations. Comput. Methods Appl. Mech. Eng. **197**, 22–35 (2007)
28. Li, J., He, Y., Chen, Z.: Performance of several stabilized finite element methods for the Stokes equations based on the lowest equal-order pairs. Computing **86**, 37–51 (2009)
29. Li, J., Shen, L., Chen, Z.: Convergence and stability of a stabilized finite volume method for the stationary Navier–Stokes equations. BIT Numer. Math. **50**, 823–842 (2010)
30. Li, R.: Generalized difference methods for a nonlinear Dirichlet problem. SIAM J. Numer. Anal. **24**, 77–88 (1987)

31. Li, R., Zhu, P.: Generalized difference methods for second order elliptic partial differential equations (I)-triangle grids. Numer. Math. J. Chinese Universities **2**, 140–152 (1982)
32. Li, R., Chen, Z., Wu, W.: Generalized Difference Methods for Differential Equations: Numerical Analysis of Finite Volume Methods. Marcel Dekker, New York (2000)
33. Lv, J., Li, Y.: L^2 error estimate of the finite volume element methods on quadrilateral meshes. Adv. Comput. Math. **33**, 129–148 (2010)
34. Shen, J.: On error estimates of the penalty method for unsteady Navier–Stokes equations. SIAM J. Numer. Anal. **32**, 386–403 (1995)
35. Temam, R.: Navier–Stokes Equations. North-Holland, Amsterdam (1984)
36. Thomée, V.: Galerkin finite element methods for parabolic problems. Lecture Notes in Math., vol. 1054, Springer-Verlag, Berlin (1984)
37. Wang, J., Wang, Y., Ye, X.: A new finite volume method for the stokes problems. Int. J. Numer. Anal. Model. **7**, 281–302 (2009)
38. Xu, J., Zhu, Y., Zou, Q.: New adaptive finite volume methods and convergence analysis. (submitted)
39. Xu, J., Zou, Q.: Analysis of linear and quadratic finite volume methods for elliptic equatiosns. Numer. Math. **111**, 469–492 (2009)
40. Ye, X.: On the relationship between finite volume and finite element methods applied to the Stokes equations. Numer. Methods Partial Differ. Equ. **5**, 440–453 (2001)