PRISMATIC MIXED FINITE ELEMENTS FOR SECOND ORDER ELLIPTIC PROBLEMS (1)

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ABSTRACT - In this paper, three families of mixed finite elements based on prisms are introduced. These spaces are analogues to those based on simplices and cubes in three space variables. Error estimates in L^2 and H^{-s} are given.

1. Introduction

We introduce three families of spaces of mixed finite elements over prisms to approximate the solutions of second order elliptic equations in three variables. The first family is an analogue of the space described by Nedelec [9] for three-dimensional problems, but different degrees of freedom are used and the number of these degrees is lower than required in [9]. The other two families are based on the spaces recently introduced in [1] and [2] for the same problems and lead to a much lower number of degrees of freedom than the first family.

In §2 we define the first family and introduce locally defined projections. In §3 and §4 we give the second and third families and the corresponding projections. The last section, §5, discusses very briefly some computational and other aspects of these methods. We shall apply the theory of Douglas and Roberts [8] to obtain error estimates in L^2 and H^{-s} for Dirichlet problems on a domain Ω of the

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form $\Omega = G\times[0,1]$ with $G\subset R^2$; appropriate assumptions on the regularity of ∂G will be made where needed.

Certain notations will be employed throughout the paper. A prism whose base is a triangle in the (x_1,x_2)-plane with three vertical edges parallel to the x_3-axis will be denoted by K, its boundary will be ∂K, $\underset{\sim}{n}$ will be the normal to ∂K (vectors will be represented by the mark $\sim$), and e will be a face of K. The space of polynomials of degree less than or equal to j in three variables will be written as P_j; $P_{m,n}$ is the space of polynomials of degree m in the two variables x_1 and x_2 and of degree n in the variable x_3; $Q_{m,n}$ is the space of polynomials of two variables (x_1,x_2) of degrees m and n in variables x_1 and x_2, respectively. Denote by $\underset{\sim}{P_j}(K)$ the vector analogue of $P_j(K)$ consisting of three copies of $P_j(K)$. Let $(\cdot,\cdot)_K$ indicate the inner product in $L^2(K)$ and $<\cdot,\cdot>_e$ that in $L^2(e)$.

We shall use the elementary differential operators

$$\underset{\sim}{\nabla}_{(x_1,x_2)}\varphi = \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}\right),$$

$$\underset{\sim}{\nabla}\varphi = \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}\right),$$

$$\mathrm{div}_{(x_1,x_2)}\underset{\sim}{V} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2},$$

$$\mathrm{div}\underset{\sim}{V} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3},$$

$$\underset{\sim}{\mathrm{curl}}_{(x_1,x_2)}\varphi = \left(\frac{\partial\varphi}{\partial x_2}, -\frac{\partial\varphi}{\partial x_1}\right),$$

$$\underset{\sim}{\mathrm{curl}}\,\underset{\sim}{V} = \left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}, \frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}, \frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right).$$

2. Nedelec-type Prismatic Mixed Finite Elements

We now introduce a family of mixed finite elements on the prism K that will be analogous to those of Nedelec [9].

Definition 1. Let

(1) $\underset{\sim}{V}(j,K) = \{\underset{\sim}{p}=(p_1,p_2,p_3):(p_1,p_2) \in P_{j,j-1}(K)^2,\ p_3 \in P_{j-1,j}(K)\}$,

(2) $W(j,K) = P_{j-1,j-1}(K)$.

Let

$$B_{j+1,j-1}(K) = \{p \in P_{j+1,j-1}(K) : p|_e \equiv 0 \text{ on the three vertical faces}\},$$

and set

$$\underset{\sim}{H}_p(\mathrm{div},K) = \{\underset{\sim}{v} \in \underset{\sim}{L}^p(K) : \mathrm{div}\,\underset{\sim}{v} \in L^2(K)\},$$

where p is any fixed number greater than two. Then, when K has flat faces, we define Π^j: $\underset{\sim}{H}_p(\mathrm{div},K) \to \underset{\sim}{V}(j,K)$ by

(2.1a) $\langle (\underset{\sim}{p}-\Pi^j\underset{\sim}{p}) \cdot \underset{\sim}{n}_e, q \rangle_e = 0$, $q \in P_{j-1}(e)$, for the two horizontal faces,

(2.1b) $\langle (\underset{\sim}{p}-\Pi^j\underset{\sim}{p}) \cdot \underset{\sim}{n}_e, q \rangle_e = 0$, $q \in Q_{j,j-1}(e)$, for the three vertical faces,

(2.1c) $(p_3-(\Pi^j\underset{\sim}{p})_3, q_3)_K = 0$, $q_3 \in P_{j-1,j-2}(K)$,

(2.1d) $((p_1,p_2)-(\Pi^j\underset{\sim}{p})_{1,2}, \underset{\sim}{\nabla}_{(x_1,x_2)}w)_K = 0$, $w \in P_{j-1,j-1}(K)$,

(2.1e) $((p_1,p_2)-(\Pi^j\underset{\sim}{p})_{1,2}, \underset{\sim}{\mathrm{curl}}_{(x_1,x_2)}q)_K = 0$, $q \in B_{j+1,j-1}(K)$,

where we indicate the third component and the first two components of $\Pi^j\underset{\sim}{p}$ by $(\Pi^j\underset{\sim}{p})_3$ and $(\Pi^j\underset{\sim}{p})_{1,2}$, respectively.

Note that, by [4], the relations (2.1a) and (2.1b) are well-defined.

Theorem 1. *The projection Π^j given by Definition 1 is unisolvent and conforming in the space $\underset{\sim}{H}_p(\mathrm{div},K)$. Moreover, if Π^{*j} is the L^2-projection on $P_{j-1,j-1}(K)$, then*

(2.2) $\Pi^{*j}\mathrm{div}\underset{\sim}{p} = \mathrm{div}\Pi^j\underset{\sim}{p}$, $\forall \underset{\sim}{p} \in \underset{\sim}{H}_p(\mathrm{div},K)$.

Proof. First, note that $\dim(\underset{\sim}{V}(j,K))=j(j+1)(j+2)+j(j+1)^2/2$. Then, the number of degrees of freedom of types (2.1a),...,(2.1e) are $j(j+1)$, $3j(j+1)$, $(j-1)j(j+1)/2$,

$j^2(j+1)/2-j$, and $j^2(j-1)/2$, respectively, so that the number of degrees of freedom is exactly the dimension of the space $\underset{\sim}{V}(j,K)$. Thus, to show the existence of Π^j it suffices to prove that a vector in $\underset{\sim}{V}(j,K)$ having vanishing degrees of freedom must itself vanish when K is the reference prism having vertices (0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,0,1) and (0,1,1).

If the degrees of freedom on the faces are zero, then

$$(2.3)\qquad \underset{\sim}{p}\cdot\underset{\sim}{n} = 0.$$

(This proves the conformity in $\underset{\sim}{H}_p(\mathrm{div},K)$, assuming the unisolvence to be demonstrated below). It follows from (2.3) that

$$p_3 = x_3(1-x_3)q_3,\ q_3 \in P_{j-1,j-2},$$

and (2.1c) implies that

$$(2.4)\qquad p_3 \equiv 0.$$

Using Green's formula and the degrees of freedom of type (2.1d), we have

$$\int_K (\mathrm{div}\,\underset{\sim}{p})^2 dx = -\int_K \underset{\sim}{p}\cdot\underset{\sim}{\nabla}\mathrm{div}\,\underset{\sim}{p}\,dx + \int_{\partial K} \underset{\sim}{p}\cdot\underset{\sim}{n}\,\mathrm{div}\,\underset{\sim}{p}\,d\gamma = 0,$$

so that

$$(2.5)\qquad \mathrm{div}_{(x_1,x_2)}\underset{\sim}{p} \equiv 0.$$

Thus, there exists $\varphi \in P_{j+1,j-1}$ such that

$$(2.6)\qquad p_1 = \frac{\partial\varphi}{\partial x_2},\ p_2 = -\frac{\partial\varphi}{\partial x_1}.$$

It follows from (2.3) for the vertical faces that we can take $\varphi=0$ on the three vertical faces, so that $\varphi \in B_{j+1,j-1}$; then (2.1e) implies that

$$p_1 \equiv 0 \text{ and } p_2 \equiv 0,$$

which with (2.4) means that $\underset{\sim}{p}\equiv 0$. Thus, unisolvence is established.

Again using Green's formula, we see that

$$(2.7)\qquad \int_K \operatorname{div}(\underset{\sim}{p}-\Pi^j\underset{\sim}{p})\varphi\, dx = -\int_K (\underset{\sim}{p}-\Pi^j\underset{\sim}{p})\cdot \underset{\sim}{\operatorname{grad}}\, \varphi\, dx + \int_{\partial K} (\underset{\sim}{p}-\Pi^j\underset{\sim}{p})\cdot \underset{\sim}{n}\varphi\, d\gamma.$$

The degrees of freedom are such that the righthand side is zero when φ is in $P_{j-1,j-1}$. Thus, (2.2) holds, and the proof is complete.

Now, let Ω be a bounded domain in R^3 and let K be a boundary prism with one curved, Lipschitz face lying in the boundary of Ω. It is necessary to modify the definition of Π^j on such prisms. Let $\underset{\sim}{V}(j,K)$ be exactly as above for the ordinary prism. Now, let $\Pi^j:\underset{\sim}{H}_p(\operatorname{div},K)\rightarrow\underset{\sim}{V}(j,K)$ be determined by the relations

$$(2.8a)\qquad < (\underset{\sim}{p}-\Pi^j\underset{\sim}{p})\cdot \underset{\sim}{n}_e, q >_e = 0,\ f \in P_{j-1}(e) \text{ for each flat horizontal face,}$$

$$(2.8b)\qquad < (\underset{\sim}{p}-\Pi^j\underset{\sim}{p})\cdot \underset{\sim}{n}_e, q >_e = 0,\ q \in Q_{j,j-1}(e) \text{ for each flat vertical face,}$$

$$(2.8c)\qquad (\operatorname{div}(\underset{\sim}{p}-\Pi^j\underset{\sim}{p}),w)_K = 0,\ w \in P_{j-1,j-1},$$

$$(2.8d)\qquad (\underset{\sim}{p}-\Pi^j\underset{\sim}{p},\underset{\sim}{v})_K = 0,\ \underset{\sim}{v} \in \{\underset{\sim}{u} \in \underset{\sim}{V}(j,K): \operatorname{div}\underset{\sim}{u}=0 \text{ and } \underset{\sim}{u}\cdot\underset{\sim}{n}_e = 0 \text{ for each flat face}\}.$$

Again, it is easy to see that Π^j is uniquely determined by (2.8) and that (2.2) holds for boundary prisms.

We have now the local properties of our spaces $\underset{\sim}{V}(j,K)$ and $W(j,K)$. To construct the spaces globally, we allow a boundary prism K to have one curved face, which we shall assume Lipschitz. Let $\{K\}=\mathcal{T}_h$ be a decomposition of the domain Ω into nonoverlapping prisms such that

(2.9a) the intersection of two distinct K's in $\mathcal{T}_h$ is either a face, an edge, a vertex, or void;

(2.9b) if $K\subset\Omega$, K has flat faces;

(2.9c) if $\operatorname{diam}(K) = h_k$, $h_k\leq h$;

(2.9d) if r_k is the radius of the ball inscribed in K, $h_k/r_k\leq$constant.

Set

(2.10a) $\underset{\sim}{V}_h = \underset{\sim}{V}_h^j = \{\underset{\sim}{u} \in \underset{\sim}{H}(\mathrm{div},\Omega): \underset{\sim}{u}|_K \in \underset{\sim}{V}(j,K),\ K \in \mathscr{T}_h\},$

(2.10b) $W_h = W_h^j = \{w \in L^2(\Omega): w|_K \in W(j,K),\ K \in \mathscr{T}_h\},$

(2.10c) $M_h = \underset{\sim}{V}_h \times W_h.$

Extend the projections Π^j and Π^{*j} to $\underset{\sim}{H}_p(\mathrm{div},\Omega)$ and $L^2(\Omega)$, respectively, as follows:

(2.11a) $\Pi_h = \Pi_h^j: \underset{\sim}{H}_p(\mathrm{div},\Omega) \rightarrow \underset{\sim}{V}_h$ satisfies $\Pi_h|_{\underset{\sim}{H}_p(\mathrm{div},K)} = \Pi^j,$

(2.11b) $\Pi^*_h = \Pi_h^{*j}: L^2(\Omega) \rightarrow W_h$ satisfies $\Pi^*_h|_{L^2(K)} = \Pi^{*j}.$

The following property of Π_h and Π^*_h results from the local property (2.2):

(2.12) $\mathrm{div}\Pi_h = \Pi^*_h\mathrm{div} : \underset{\sim}{H}_p(\mathrm{div},\Omega) \rightarrow W_h.$

The approximation properties of Π_h and Π^*_h can also be seen directly from their local properties:

(2.13a) $\|\underset{\sim}{\psi}-\Pi_h\underset{\sim}{\psi}\|_0 \leqslant c\|\underset{\sim}{\psi}\|_r h^r,\ \underset{\sim}{\psi} \in \underset{\sim}{H}^r(\Omega),\ 1\leqslant r\leqslant j,$

(2.13b) $\|w-\Pi^*_h w\|_{-s} \leqslant c\|w\|_r h^{r+s},\ w \in H^r(\Omega),\ 0\leqslant r,\ s\leqslant j.$

3. Prismatic BDDF Mixed Finite Elements

DEFINITION 2. Let

(1) $\underset{\sim}{V}(j,K) = \underset{\sim}{P}_j(K) + \mathrm{Span}[\underset{\sim}{\mathrm{curl}}(x_2^{j+1}x_3,0,0),$

$\underset{\sim}{\mathrm{curl}}(x_2x_3^{j+1},-x_1x_3^{j+1},0);$

$\underset{\sim}{\mathrm{curl}}(0,x_1^{i+1}x_2^{j-i}x_3,0);\ i=1,\ldots,j];$

(2) $W(j,K) = P_{j-1}(K).$

LEMMA 1. *The* $j+2$ *polynomial vectors of degree* $j+1$ *added to* $\underset{\sim}{P}_j(K)$ *in Definition 2 are linearly independent.*

Proof. Suppose that

$$\sum_{i=1}^{j} c_i(-x_1^{i+1}x_2^{j-i},0,(i+1)x_1^i x_2^{j-i}x_3) + a(0,x_2^{j+1},-(j+1)x_2^j x_3)$$

$$+ b(j+1)x_1x_3^j, (j+1)x_2x_3^j, -2x_3^{j+1}) = 0.$$

Summing on the first component shows that $b=0$ and $c_1=...=c_j=0$. Then, the vanishing of the remaining term implies that $a=0$.

EXAMPLE. *The space* $\underset{\sim}{V}(1,K)$ *has dimension 15 and consists of* $\underset{\sim}{P}_1(K)$ *plus the span of the three vectors* $(0,x_2^2,-2x_2x_3)$, $(2x_1x_2,2x_2x_3,-2x_3^2)$, *and* $(-x_1^2,0,2x_1x_3)$.

Let

$$B_{j+1}(K) = \{\xi \in P_{j+1}(K): \xi|_e \equiv 0 \text{ on the three vertical faces}\}.$$

Let $p>2$. Then, when K has flat faces, we define $\Pi^j:\underset{\sim}{H}_p(\text{div},K)\rightarrow\underset{\sim}{V}(j,K)$ by

(3.1a) $\quad < (\underset{\sim}{\psi}-\Pi^j\underset{\sim}{\psi}) \cdot \underset{\sim}{n}_e, q >_e = 0,\ q \in P_j(e)$, for the five flat faces,

(3.1b) $\quad (\psi_3-(\Pi^j\underset{\sim}{\psi})_3,q_3)_K = 0,\ q_3 \in P_{j-2}(K)$,

(3.1c) $\quad ((\psi_1,\psi_2)-(\Pi^j\underset{\sim}{\psi})_{1,2},\underset{\sim}{\nabla}_{(x_1,x_2)}w)_K = 0,\ w \in P_{j-1}(K)$,

(3.1d) $\quad ((\psi_1,\psi_2)-(\Pi^j\underset{\sim}{\psi})_{1,2},\ \underset{\sim}{\text{curl}}_{(x_1,x_2)}q)_K = 0,\ q \in B_{j+1}(K)$.

THEOREM 2. *The finite element given by Definition 2 is unisolvent and conforming in the space* $\underset{\sim}{H}(\text{div},K)$. *Moreover, if* Π^{*j} *is the* L^2*-projection on* $W(j,K)$,

(3.2) $\quad \Pi^{*j}\text{div}\underset{\sim}{\psi} = \text{div}\Pi^j\underset{\sim}{\psi},\ \underset{\sim}{\psi} \in \underset{\sim}{H}_p(\text{div},K)$.

Proof. Again the total number of degrees of freedom is equal to the dimension of $\underset{\sim}{V}(j,K)$, since $\dim(\underset{\sim}{V}(j,K))=(j^3+6j^2+13j+10)/2$, while

$$\#\text{dof}(3.1a) = 5(j+1)(j+2)/2,$$

$$\#\text{dof}(3.1b) = (j-1)j(j+1)/6,$$

$$\#\text{dof}(3.1c) = j(j+1)(j+2)/6-j,$$

$$\#\text{dof}(3.1d) = (j-1)j(j+1)/6.$$

As above, to show the existence of Π^j it suffices to prove uniqueness on the reference prism. Let all degrees of freedom vanish. Then, the components of $\underset{\sim}{\psi}$ are of the form

$$\psi_1 = -\sum_{i=1}^{j} c_i x_1^{i+1} x_2^{j-i} + b(j+1)x_1 x_3^j + r_1,$$

$$\psi_2 = a x_2^{j+1} + b(j+1)x_2 x_3^j + r_2,$$

$$\psi_3 = -a(j+1)x_2^j x_3 - 2b x_3^{j+1} + \sum_{i=1}^{j} c_i (i+1) x_1^i x_2^{j-i} x_3 + r_3,$$

where $r_i \in P_j(K)$. Then, (3.1a) applied on $x_1=0$ implies that

$$r_1 = x_1 s_1,\ s_1 \in P_{j-1}(K).$$

Similarly, (3.1a) applied on $x_2=0$, $x_3=1$, and $x_1+x_2=1$ shows that

$$r_2 = x_2 s_2,\ s_2 \in P_{j-1}(K),$$

$$r_3 = x_3 s_3,\ s_3 \in P_{j-1}(K),$$

$$a = 0,\ c_1 = \dots = c_j = 0,\ r_3 = x_3(1-x_3)q_3,\ q_3 \in P_{j-2},$$

$$b = 0.$$

Hence, (3.1b) shows that

$$\psi_3 = 0. \tag{3.3}$$

Using Green's formula and the degrees of freedom of type (3.1c), we have

$$\int_K (\operatorname{div} \underset{\sim}{\psi})^2 dx = -\int_K \underset{\sim}{\psi} \cdot \underset{\sim}{\nabla} \operatorname{div} \underset{\sim}{\psi}\, dx + \int_{\partial K} \underset{\sim}{\psi} \cdot \underset{\sim}{n} \operatorname{div} \underset{\sim}{\psi}\, d\gamma = 0,$$

so that

$$\operatorname{div}_{(x_1,x_2)} \underset{\sim}{\psi} = 0. \tag{3.4}$$

As for the first family, there exists $\varphi \in P_{j+1}$ such that

$$\psi_1 = \frac{\partial \varphi}{\partial x_2},\ \psi_2 = -\frac{\partial \varphi}{\partial x_1}. \tag{3.5}$$

It follows from $\underset{\sim}{\psi} \in \underset{\sim}{P}_j(K)$ and (3.1a) applied on the vertical faces that φ can be taken to vanish on the three vertical faces, so that $\varphi \in B_{j+1}(K)$. Then (3.1d) implies that

$$\psi_1 = 0 \text{ and } \psi_2 = 0,$$

which with (3.3) means that $\underset{\sim}{\psi}=0$.

Again by Green's formula, we have

$$\int_K \operatorname{div}(\underset{\sim}{\psi}-\Pi^j\underset{\sim}{\psi})\varphi \, dx = -\int_K (\underset{\sim}{\psi}-\Pi^j\underset{\sim}{\psi}) \cdot \underset{\sim}{\nabla} \varphi \, dx + \int_{\partial K} (\underset{\sim}{\psi}-\Pi^j\underset{\sim}{\psi}) \cdot \underset{\sim}{n} \, \varphi \, d\gamma. \tag{3.6}$$

The degrees of freedom are such that the right-hand side is zero when $\varphi \in P_{j-1}(K)$. This completes the proof.

Boundary elements are allowed to have one curved face. The projection Π^j can be defined in a way analogous to (2.8):

$$< (\underset{\sim}{\psi}-\Pi^j\underset{\sim}{\psi}) \cdot \underset{\sim}{n}_e, q >_e = 0, \; q \in P_{j-1}(e), \text{ for each flat face,} \tag{3.7a}$$

$$(\operatorname{div}(\underset{\sim}{\psi}-\Pi^j\underset{\sim}{\psi}), w)_K = 0, \; w \in P_{j-1}(K), \tag{3.7b}$$

$$(\underset{\sim}{\psi}-\Pi^j\underset{\sim}{\psi}, \underset{\sim}{v})_K = 0, \; \underset{\sim}{v} \in \{\underset{\sim}{u} \in \underset{\sim}{V}(j,K): \operatorname{div} \underset{\sim}{u}=0 \text{ and } \underset{\sim}{u} \cdot \underset{\sim}{n}_e = 0 \text{ for each flat face}\}. \tag{3.7c}$$

It is clear that Π^j is defined on boundary elements.

Let $\mathcal{T}_h=\{K\}$ be a decomposition of Ω into prisms satisfying (2.9). Construct global projections $\Pi_h:\underset{\sim}{H}_p(\operatorname{div},\Omega)\rightarrow\underset{\sim}{V}_h$ and $\Pi^*_h:L^2(\Omega)\rightarrow W_h$ by piecing together the appropriate Π^j's and Π^{*j}'s, respectively. The following approximation properties of the projections are easily seen from the local nature of their definitions:

$$\|\underset{\sim}{\psi}-\Pi_h\underset{\sim}{\varphi}\|_0 \leq c\|\underset{\sim}{\psi}\|_s \, h^s, \; \underset{\sim}{\psi} \in \underset{\sim}{H}^s(\Omega), \; 1\leq s\leq j+1. \tag{3.8a}$$

$$\|w-\Pi^*_h w\|_{-s} \leq c\|w\|_s h^{s+r}, \; w \in H^s(\Omega), \; 1\leq r, \; s\leq j. \tag{3.8b}$$

4. Prismatic BDFM Mixed Finite Elements

Denote by $P_j(\text{hom},x_1,x_2)$ the homogeneous polynomials of degree j in the variables x_1 and x_2. Then we introduce the third family of mixed finite elements:

DEFINITION 3. Let

(1) $\underset{\sim}{V}(j,K) = \{\underset{\sim}{\psi}=(\psi_1,\psi_2,\psi_3):(\psi_1,\psi_2) \in [P_j\backslash\{x_3^j\}]^2|_K,$

$\psi_3 \in [P_j\backslash P_j(\mathrm{hom},x_1,x_2)]|_K\},$

(2) $W(j,K) = P_{j-1}(K).$

EXAMPLE. *The space* $\underset{\sim}{V}(1,K)$ *has dimension eight and consists of the polynomials*

$$\psi_1 = a_0+a_1+a_2x_2,$$

$$\psi_2 = b_0+b_1x_1+b_2x_2,$$

$$\psi_3 = c_0+c_3x_3.$$

Let $p>2$ and let K have flat faces. Then we define $\Pi^j:\underset{\sim}{H}_p(\mathrm{div},K)\rightarrow\underset{\sim}{V}(j,K)$ by

(4.1a) $<(\underset{\sim}{\psi}-\Pi^j\underset{\sim}{\psi})\cdot\underset{\sim}{n}_e, q>_e = 0,\ q \in P_{j-1}(e)$ for the two horizontal faces,

(4.1b) $<(\underset{\sim}{\psi}-\Pi^j\underset{\sim}{\psi})\cdot\underset{\sim}{n}_e, q>_e = 0,\ q \in P_j\backslash\{x_3^j\}|_e,$ for the three vertical faces,

(4.1c) $(\psi_3-(\Pi^j\underset{\sim}{\psi})_3, q_3)_K = 0,\ q_3 \in P_{j-2}(K),$

(4.1d) $((\psi_1,\psi_2)-(\Pi^j\underset{\sim}{\psi})_{1,2}, \underset{\sim}{\nabla}_{(x_1,x_2)}w)_K = 0,\ w \in P_{j-2}(K),$

(4.1e) $((\psi_1,\psi_2)-(\Pi^j\underset{\sim}{\psi})_{1,2}, \underset{\sim}{\mathrm{curl}}_{(x_1,x_2)}q)_K = 0,\ q \in B_{j+1}(K).$

THEOREM 3. *The vector element given in Definition 3 is unisolvent and conforming in the space* $\underset{\sim}{H}(\mathrm{div},K)$. *Moreover, if* Π^{*j} *is the* L^2*–projection on* $W(j,K)$,

(4.2) $\Pi^{*j}\mathrm{div}\underset{\sim}{\psi} = \mathrm{div}\ \Pi^j\underset{\sim}{\psi},\ \forall\underset{\sim}{\psi} \in \underset{\sim}{H}_p(\mathrm{div},K).$

Proof. It can be seen that the dimension of $\underset{\sim}{V}(j,K)$ and the number of degrees of freedom are $(j^3+6j^2+9j)/2$. Then, it follows from (4.1a-b) that

(4.3) $\underset{\sim}{\psi}\cdot\underset{\sim}{n} = 0,$

which proves the conformity in $\underset{\sim}{H}(\mathrm{div},K)$. The proofs of unisolvence and of the expression of (4.2) are similar to those of Theorem 1.

Boundary elements are again allowed to have at most one curved face. The projection Π^j can be defined for such elements similarly:

(4.4a) $\quad < (\underset{\sim}{\psi}-\Pi^j\underset{\sim}{\psi}) \cdot \underset{\sim}{n}_e, q >_e = 0,\ q \in P_{j-1}(e)$, for each flat horizontal face,

(4.4b) $\quad < (\underset{\sim}{\psi}-\Pi^j\underset{\sim}{\psi}) \cdot \underset{\sim}{n}_e, q >_e = 0,\ q \in P_j\backslash\{x_3^j\}|_e$, for each flat vertical face,

(4.4c) $\quad (\mathrm{div}(\underset{\sim}{\psi}-\Pi^j\underset{\sim}{\psi}),w)_K = 0,\ w \in P_{j-1}(K)$,

(4.4d) $\quad (\underset{\sim}{\psi}-\Pi^j\underset{\sim}{\psi},\underset{\sim}{v})_K = 0,\ \underset{\sim}{v} \in \{\underset{\sim}{u} \in \underset{\sim}{V}(j,K): \mathrm{div}\,\underset{\sim}{u}= 0$

and $\underset{\sim}{u} \cdot \underset{\sim}{n}_e = 0$ for each flat face$\}$.

The approximation properties for the globally defined Π_h and Π^*_h are the same as given in (2.13).

5. The Dirichlet Problem

Consider the Dirichlet problem

(5.1a) $\quad Lu = f$, in Ω,

(5.1b) $\quad u = -g$, on $\partial\Omega$,

where $Lu=-\mathrm{div}(a(x)\underset{\sim}{\mathrm{grad}}\,u)$, $\Omega=G\times[0,1]$ with $G\subset R^2$ and ∂G being smooth, and $a(x)$ is a smooth, positive function on the closure of Ω. We assume that the problem (5.1) has a periodic solution in the direction of x_3 of period Ω. The periodicity assumption is made to permit the application of the general argument of Douglas and Roberts [8] to obtain optimal order estimates in Sobolev spaces of negative index by means of duality; it is not needed for the L^2 estimates stated below.

Let

(5.2) $\quad \underset{\sim}{q} = -a\,\underset{\sim}{\mathrm{grad}}\,u,\ c(x) = a(x)^{-1}$,

and factor (5.1a) into the first order system

(5.3a) $c\underset{\sim}{q} + \underset{\sim}{\text{grad}}\, u = 0,$

(5.3b) $\text{div}\, \underset{\sim}{q} = f,$

for $x \in \Omega$. The weak form of (5.3) and (5.1b) appropriate for mixed finite element methods is given by seeking $\{\underset{\sim}{q}, u\} \in \underset{\sim}{H}(\text{div},\Omega)\times L^2(\Omega)$ such that

(5.4a) $(c\underset{\sim}{q}, \underset{\sim}{v}) - (\text{div}\, \underset{\sim}{v}) = \, < g, \underset{\sim}{v} \cdot \underset{\sim}{n} >, \; \underset{\sim}{v} \in \underset{\sim}{H}(\text{div},\Omega),$

(5.4b) $(\text{div}\underset{\sim}{q}, w) = (f,w), \; w \in L^2(\Omega).$

Let $\mathscr{T}_h$ be a decomposition of Ω into prisms and assume that $\mathscr{T}_h$ satisfies (2.9). The mixed finite element approximation $\{\underset{\sim}{q}_h, u_h\} \in \underset{\sim}{V}_h\times W_h$ is defined as the solution of the equations

(5.5a) $(c\, \underset{\sim}{q}_h, \underset{\sim}{v}) - (\text{div}\underset{\sim}{v}, u_h) = \, < g, \underset{\sim}{v} \cdot \underset{\sim}{n} >, \; \underset{\sim}{v} \in \underset{\sim}{V}_h,$

(5.5b) $(\text{div}\, \underset{\sim}{q}_h, w) = (f,w), \; w \in W_h.$

The existence and uniqueness of $\{\underset{\sim}{q}_h, u_h\}$ follow from the general argument of Douglas and Roberts [8]. Moreover, the error analysis of [3] applies without modification to any of three spaces $M_h = \underset{\sim}{V}_h \times W_h$ corresponding to the above definitions, since the derivation of the estimates depends only on the properties of the projections Π_h and Π^*_h and the regularity of G. Note that the general argument of [6] shows that one extra derivative is required of u for $s=j-1$ estimate and two for $s=j$. We now state two theorems as follows.

THEOREM 4. *Let* $M_h = \underset{\sim}{V}_h \times W_h$ *be determined by either Definition 1 or by Definition 3, and let* $\{\underset{\sim}{q}_h, u_h\} \in M_h$ *be the solution of the mixed finite element method (5.5). Then,*

$$\|u-u_h\|_{-s} \leqslant \begin{cases} c\|u\|_r h^{r+s}, & 0\leqslant s\leqslant j-2,\ 2\leqslant r\leqslant j, \\ c\|u\|_{r+1} h^{r+s}, & s=j-1,\ 1\leqslant r\leqslant j, \\ c\|u\|_{r+2} h^{r+s}, & s=j,\ 0\leqslant r\leqslant j; \end{cases}$$

$$\|\underset{\sim}{q}-\underset{\sim}{q}_h\|_{-s} \leq \begin{cases} c\|u\|_{r+1}h^{r+s}, & 0\leq s\leq j-1,\ 1\leq r\leq j, \\ c\|u\|_{r+2}h^{r+j}, & s=j,\ 0\leq r\leq j; \end{cases}$$

$$\|\mathrm{div}(\underset{\sim}{q}-\underset{\sim}{q}_h)\|_{-s} \leq c\|u\|_{r+2}h^{r+s},\ 0\leq s\leq j,\ 0\leq r\leq j.$$

Moreover,

$$\|u_h-\Pi^*_h u\|_0 \leq c\|u\|_{j+2}h^{j+1}.$$

THEOREM 5. *Let* $M_h=\underset{\sim}{V}_h\times W_h$ *be determined by Definition 2, and let* $\{\underset{\sim}{q}_h,u_h\}\in M_h$ *be the solution of the mixed finite element method (5.5). Then,*

$$\|u-u_h\|_{-s} \leq c\|u\|_r h^{r+s},\ 2\leq r\leq j,\ 0\leq s\leq j,$$

$$\|\underset{\sim}{q}-\underset{\sim}{q}_h\|_{-s} \leq c\|u\|_{r+1}h^{r+s},\ 1\leq r\leq j+1,\ 0\leq s\leq j-1,$$

$$\|\mathrm{div}(\underset{\sim}{q}-\underset{\sim}{q}_h)\|_{-s} \leq c\|\mathrm{div}\ \underset{\sim}{q}\|_r h^{r+s},\ 0\leq r\leq j,\ 0\leq s\leq j.$$

Moreover,

$$\|u_h-\Pi^*_h u\|_0 \leq c\|u\|_{j+2}h^{\min(j+2,2j)}.$$

We have seen that the last two families have a much lower number of degrees of freedom than the first family for corresponding rates of convergence. We can post-process the approximate solution as in [3], [1], or [2] to obtain better approximations for the scalar filed u. Hybridization of the mixed method, alternating-direction iterative techniques, and superconvergence can be considered in the same manner as in the papers [1], [2], [7], [6], and [5].

REFERENCES

[1] F. BREZZI, J. DOUGLAS JR., R. DURÁN, M. FORTIN, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math., 51 (1987), 237-250.

[2] F. BREZZI, J. DOUGLAS JR., M. FORTIN, L. D. MARINI, *Efficient rectangular mixed finite elements in two and three space variables*, R.A.I.R.O. Modélisation Mathématique et Analyse Numérique, 21 (1987), 581-604.

[3] F. BREZZI, J. DOUGLAS JR., L. D. MARINI, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math. 47 (1985), 217-235.

[4] F. BREZZI, L. D. MARINI, P. PIETRA, *Two-dimensional exponential fitting and applications to semiconductor device equations*, SIAM J. Numer. Anal., to appear.

[5] J. DOUGLAS JR., R. DURÁN, P. PIETRA, *Alternating-direction iteration for mixed finite element methods*, Computing Methods in Applied Science and Engineering, VII, to appear.

[6] J. DOUGLAS JR., R. DURÁN, P. PIETRA, *Formulation of alternating-direction iterative methods for mixed methods in three space*, in Numerical Approximation of Partial Differential Equations, 1987, E. L. Ortiz, ed., North-Holland, Amsterdam, 21-30.

[7] J. DOUGLAS JR., P. PIETRA, *A description of some alternating-direction iterative techniques for mixed finite element methods*, in Mathematical and Computational Methods in Seismic Exploration and Reservoir Modeling, 1986, W. E. Fitzgibbon, ed., SIAM, Philadelphia, 37-53.

[8] J. DOUGLAS JR., J. E. ROBERTS, *Global estimates for mixed methods for second order elliptic problems*, Math. Comp., 45 (1985), 39-52.

[9] J. C. NEDELEC, *A new family of mixed finite elements in R^3*, Numer. Math., 50 (1986), 57-82.