STABILITY AND CONVERGENCE OF A FINITE ELEMENT METHOD FOR REACTIVE TRANSPORT IN GROUND WATER

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Abstract. An explicit finite element method is used to solve the linear convection–diffusion-reaction equations governing contaminant transport in ground water flowing through an adsorbing porous medium. The use of discontinuous finite elements for the convective part of the equations combined with mixed finite elements for the diffusive part renders the method for the concentration solution, which displays strong gradients, trivially conservative and fully parallelizable. We carry out a stability and convergence analysis. In particular, the method is proven to satisfy a maximum principle, to be total variation bounded, and to converge to the unique weak solution of the equations. Special attention is paid to the convective part of the equations. Numerical simulations are presented and discussed.

Key words. convection–diffusion-reaction equation, finite element and volume method, conservation law, stability, convergence, mixed method

AMS subject classifications. 65N30, 65N10, 76S05, 76T05

1. Introduction. In this paper we propose and analyze a finite element method for solving the linear convection–diffusion-reaction equation

\[ \frac{\partial}{\partial t} (\Phi u) + \text{div}(Vu - D\nabla u) = -Ku, \]

which describes the transport of a solute in a fluid phase flowing through a porous medium [1], [16]. In this case, \( u = u(t, x, y) \) is the concentration of the solute in the fluid phase for which we solve (1.1), \( V = V(t, x, y) \) is the Darcy velocity of fluid, \( \Phi \) is the volume fraction-dependent constant, \( D \) is the diffusion constant, and \( K = K(t, x, y) \geq 0 \) is the first-order chemical reaction rate. This equation, while formally parabolic, is more nearly hyperbolic in practice [4]. In recent years many finite element methods have been proposed to solve this important partial differential equation. The classes of optimal spatial methods and characteristic methods have been extensively studied in [2], [9], [15], [17], [18], for example. However, all these finite element methods are defined by taking advantage of the parabolicity of the equation for the concentration \( u \). As a result, the solution of the differential equation is required to be very smooth in the derivation of error estimates, and the constants for the error estimates blow up as the coefficient of the diffusion term goes to zero.

In this paper we propose and analyze a finite element method for numerically solving (1.1). It is similar to a finite element method introduced in [5], [3], [6], [10], [11] in that we approximate the convective part of the equation using an upwinding discontinuous finite element method or an upwinding finite volume method [20], [19]. We use, however, a mixed finite element method for the diffusive part of (1.1) [8].
main advantages of this method are that it is trivially conservative and fully parallelizable and that it can capture discontinuities within a couple of elements without producing spurious oscillations.

A stability and convergence analysis is carried out here for the finite element method for (1.1) in two space dimensions. While a stability analysis was completed for the similar approach for the two-dimensional semiconductor device equations in [6], here we are able to prove much stronger results than those obtained in [6]. Namely, besides a strong maximum principle, the boundedness of the total variation and the modulus of continuity in time of the approximate solution is proven here; only an estimate of the weak derivatives of the approximate solution is given in [6]. These properties suffice to show that the numerical method converges to the weak solution of the differential equation; in [6], however, convergence of the approximate solutions to the weak solution is proven under the assumption that there is a convergent subsequence. It is also emphasized that this paper contains the first stability analysis for the two-dimensional equation (1.1) with the diffusion term included and the first convergence analysis for (1.1) with the boundary conditions. The properties derived in this paper will be exploited in a forthcoming paper where error estimates will be obtained with minimum requirements on the solution and with the property that the constant for the error estimates does not involve the small diffusion coefficient. Especially, the error estimates apply to the case of $D$ equal to zero.

Equation (1.1) is completed by specifying the boundary and initial conditions

$\frac{\partial u}{\partial \nu} = 0$, \quad $(x, y) \in \partial \Omega_1$, \quad $t \in J$,  
\[ (1.2a) \]

$u = u_D$, \quad $(x, y) \in \partial \Omega_2$, \quad $t \in J$,  
\[ (1.2b) \]

$u(0, x, y) = u_{\text{init}}(x, y)$, \quad $(x, y) \in \Omega$,  
\[ (1.2c) \]

where $J = (0, T)$, $\Omega = (0, 1)^2$, $\partial \Omega = \partial \Omega_1 \cup \partial \Omega_2$ with $\partial \Omega_1 \cap \partial \Omega_2 = \emptyset$ and $\Omega_1$ containing the endpoints of its segments, and $\nu$ denotes the normal unit vector to $\partial \Omega$. The boundary conditions need to be modified properly in the case of $D = 0$. Namely, only the inflow boundary condition is imposed for the concentration (see (3.9) below). Moreover, in this case, note that, while (1.1) is analogous to a classical conservation law, the value of the Darcy velocity $V$ at a point $(t, x, y)$ contains the information of all the values of the solution $u(t, \cdot, \cdot)$ on $\Omega$. Hence, a perturbation of the solution $u$ at any given point of the domain has a global effect immediately. This is in sharp contrast with the classical conservation laws where local perturbations of the solution have a local effect in finite time.

The rest of the paper is organized as follows. The finite element method is defined in the next section. Then, in section 3 we state and discuss our main results on a maximum principle (Theorem 3.1), a total variation boundedness of the scheme (Theorem 3.2), continuity with respect to data (Theorem 3.3), and convergence to the weak solution (Theorem 3.4). The proofs of these properties are carried out in sections 4, 5, 6, and 7, respectively. Numerical results are displayed in section 8. These numerical results are devised to test the performance of the method and to indicate the order of convergence. Finally, a concluding remark is given in section 9.

2. The finite element method. In this section we define the finite element method for approximating the solution of the differential system (1.1). Toward that end, let $\{x_{i+1/2}\}_{i=0}^{n_x}$ and $\{y_{j+1/2}\}_{j=0}^{n_y}$ be a partition of $\Omega$ with $x_{1/2} = y_{1/2} = 0$ and $x_{n_x+1/2} = y_{n_y+1/2} = 1$ and let $\{t^n\}_{n=0}^{n_T}$ be a partition of $[0, T]$ with $t^0 = 0$ and $t^{n_T} = T$. Then, we introduce the following notation:
respectively. To discretize (1.1), we first discretize the data as follows:

\[ x_i = (x_{i-1/2} + x_{i+1/2})/2, \quad y_j = (y_{j-1/2} + y_{j+1/2})/2, \]

\[ I^x_i = (x_{i-1/2}, x_{i+1/2}), \quad I^y_j = (y_{j-1/2}, y_{j+1/2}), \]

\[ \Delta x_i = x_{i+1/2} - x_{i-1/2}, \quad \Delta y_j = y_{j+1/2} - y_{j-1/2}, \]

\[ J^n = [t^n, t^{n+1}), \quad \Delta t^n = t^{n+1} - t^n, \]

\[ \Delta x = \max_{1 \leq i \leq n_x} \Delta x_i, \quad \Delta y = \max_{1 \leq j \leq n_y} \Delta y_j, \]

\[ \Delta t = \max_{0 \leq n \leq T} \Delta t^n, \quad h = \max\{\Delta x, \Delta y\}. \]

We tacitly assume that each exterior edge has imposed on it either Dirichlet or Neumann conditions but not both. Associated with these partitions, we introduce the spaces

\[ Q_h = \{ v \in H(\text{div}; \Omega) : v |_{I^x_i \times I^y_j} = (a_{i,j}^1 + a_{i,j}^2 x, a_{i,j}^3 + a_{i,j}^4 y), \quad a_{i,j}^k \in \mathbb{R}, \]

\[ i = 1, \ldots, n_x, \quad j = 1, \ldots, n_y, \quad v \cdot |_{\partial \Omega^1} = 0 \}, \]

\[ W_h = \{ w \in L^\infty(\Omega) : w |_{I^x_i \times I^y_j} \in P^0(I^x_i \times I^y_j), \quad i = 1, \ldots, n_x, \quad j = 1, \ldots, n_y \}, \]

\[ W_{\Delta t} = \{ w \text{ right continuous} : w |_{J^n} \in P^0(J^n), \quad n = 0, \ldots, n_T - 1 \}. \]

If \( v \in Q_h, v_{i,j+1/2} \) and \( v_{i,j+1/2} \) denote \( v(x_{i+1/2}, y_j) \) and \( v(x_i, y_{j+1/2}) \), respectively. If \( w \in W_h \), then \( w_{i,j} \) represents the constant value \( w(x, y), (x, y) \in I^x_i \times I^y_j \). \( w^n \) indicates the constant \( w(t), t \in J^n \), if \( w \in W_{\Delta t} \). For notational and expositional convenience, let \( \Delta x_0 = \Delta x_1, \Delta x_{n_x+1} = \Delta x_{n_x}, \Delta y_0 = \Delta y_1, \Delta y_{n_y+1} = \Delta y_{n_y}, \Delta x_{i+1/2} = (\Delta x_i + \Delta x_{i+1})/2, \)

\[ i = 1, \ldots, n_x, \quad \Delta y_{j+1/2} = (\Delta y_j + \Delta y_{j+1})/2, \]

\[ j = 1, \ldots, n_y, \] and \( \Phi = 1. \)

Finally, define the notation \( v^+ = \max\{v, 0\} \) and \( v^- = \min\{v, 0\} \).

Let \( P_{Q_h}, P_{W_h}, \) and \( P_{W_{\Delta t}} \) denote the \( L^2 \)-projections into \( Q_h, W_h, \) and \( W_{\Delta t} \), respectively. To discretize (1.1), we first discretize the data as follows:

\[ u_{\text{init}, h} = P_{W_h} u_{\text{init}}, \]

\[ u_{D, \Delta t} = P_{W_{\Delta t}} u_D, \]

\[ V_h = P_{Q_h} V. \]

The subscript \( h \) is omitted below when no ambiguity occurs. Then, the approximate solution \( u_n \in W_{\Delta t} \otimes W_h \) is required to satisfy the equation for \( n = 0, \ldots, n_T - 1, i = 1, \ldots, n_x, \) and \( j = 1, \ldots, n_y, \)

\[ \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t^n} + \frac{f^n_{1,i+1/2,j} - f^n_{1,i-1/2,j}}{\Delta x_i} + \frac{f^n_{2,i,j+1/2} - f^n_{2,i,j-1/2}}{\Delta y_j} - \frac{D}{\Delta x_i} (q^n_{1,i+1/2,j} - q^n_{1,i-1/2,j}) - \frac{D}{\Delta y_j} (q^n_{2,i,j+1/2} - q^n_{2,i,j-1/2}) = -K^n_{i,j} u^n_{i,j}, \]

where

\[ f^n_{1,i-1/2,j} = u^n_{i-1,j} V^{n+}_{1,i-1/2,j} + u^n_{i,j} V^{n-}_{1,i-1/2,j}, \]

\[ f^n_{2,i,j-1/2} = u^n_{i-1,j} V^{n+}_{2,i,j-1/2} + u^n_{i,j} V^{n-}_{2,i,j-1/2}, \]

and the function \( q_h = (q_1, q_2) \in W_{\Delta t} \otimes Q_h \) is the solution of

\[ (q_h(t^n), v_h) = -(u_h(t^n), \text{div} v_h) + \langle u_{D, \Delta t}, v_h \cdot \nu \rangle_{\partial \Omega^D} \quad \forall v_h \in Q_h. \]
After the mass matrix has been mass lumped [22], the expression for the degrees of freedom of \( q_h \) is taken as follows:

\[
\begin{align*}
(2.2d) & \quad q^n_{1,i-1/2,j} = (u_{i,j}^n - u_{i-1,j}^n) / \Delta x_{i-1/2}, \\
(2.2e) & \quad q^n_{2,i,j-1/2} = (u_{i,j}^n - u_{i,j-1}^n) / \Delta y_{j-1/2}.
\end{align*}
\]

Finally, the Neumann boundary condition (1.2a) is discretized by the usual reflection principle, and on \( \partial \Omega_2 \) \( u_h \) is defined by \( u_{D,\Delta t} \). This implies that, if \((x_{1/2},y_j)\) lies on the Neumann boundary \( \partial \Omega_1 \), \( u^n_{0,j} \) in (2.2) and the subsequent analysis is calculated by

\[
u^n_{0,j} = u^n_{1,j};
\]

if it is on the Dirichlet boundary \( \partial \Omega_2 \), \( u^n_{0,j} \) is computed by

\[
u^n_{0,j} = u^n_{D,\Delta t}(x_{1/2},y_j).
\]

Similar extensions hold for \( u^n_{1,0} \), \( u^n_{n_x+1,j} \), and \( u^n_{i,n_y+1} \) in (2.2) and the subsequent analysis.

Note that the lowest-order Raviart–Thomas mixed method [21] over rectangles has been used in (2.2a). Since the elements in \( Q_h \) have continuous normal components on interelement edges, the numerical fluxes \( f^n_{1,i-1/2,j} \) and \( f^n_{2,i,j-1/2} \) in (2.2b) and (2.2c) are well defined. Furthermore, if appropriate approximations of the coefficient \( V_h \) are introduced and the mass-lumping technique is used as in (2.2d) and (2.2e), the conservative scheme (2.2a) can be deduced from the discontinuous finite element method [7], [12] or from the finite volume method [20], [19] combined with the mixed finite element method [22]. Finally, the scheme applies to the case of \( D = 0 \).

The following approximation properties are used later [14], [21]:

\[
\begin{align*}
(3.1a) & \quad ||V^n||_{L^\infty(\Omega)} \leq C_0 ||V^n||_{L^\infty(\Omega)}, \\
(3.1b) & \quad ||\text{div } V^n||_{BV(\Omega)} \leq C_0 ||\text{div } V^n||_{BV(\Omega)}, \\
(3.1c) & \quad u_D \in L^\infty(J; BV(\partial \Omega_2)), \\
(3.1d) & \quad u_D \in L^1(\partial \Omega_2; BV(J)), \\
(3.1e) & \quad \text{div } V \in L^\infty(J; BV(\Omega)), \\
(3.1f) & \quad K \in [0,K^*], \\
(3.1g) & \quad u_{\text{init}} \in BV(\Omega), K \in L^\infty(J; BV(\Omega)).
\end{align*}
\]

3. Stability and convergence results. In this section we state and discuss the stability and convergence results of the scheme (2.2). Let \( QT = T \times \Omega \). We assume that the data satisfy the following conditions:

\[
\begin{align*}
(3.2a) & \quad ||\text{div } V||_{L^\infty(I_T \times \Omega)} \leq C_0 (\Delta x_i + \Delta y_j)||\nabla V^n||_{L^\infty(I_T \times \Omega)}, \\
for & \quad i = 1, \ldots, n_x, j = 1, \ldots, n_y, \text{ where } C_0 \text{ is independent of } i \text{ and } j.
\end{align*}
\]
For expository convenience, let

\[ V_1^* = C_0\|V_1\|_{L^\infty(Q_T)} , \quad V_2^* = C_0\|V_2\|_{L^\infty(Q_T)} , \quad V_D^* = C_0\|\text{div } V\|_{L^\infty(Q_T)}. \]

**Theorem 3.1 (stability).** Suppose that (3.1a), (3.1b), (3.1f), and for \( n = 0, \ldots, n_T - 1 \) the following Courant–Friedrichs–Lewy (CFL) condition is satisfied:

\[ \Delta t^* \leq \frac{1}{D_{ij}^* + 2V_1^*/\Delta x_i + 2V_2^*/\Delta y_j}, \quad i = 1, \ldots, n_x, j = 1, \ldots, n_y, \]

where \( D_{ij}^* = \frac{D}{\Delta x_i} (\frac{1}{\Delta x_{i+1/2}} + \frac{1}{\Delta x_{i-1/2}}) + \frac{D}{\Delta y_j} (\frac{1}{\Delta y_{j+1/2}} + \frac{1}{\Delta y_{j-1/2}}) + K^* \). Then

\[ 0 \leq \|u_h(t, x, y) - e^{tV_0} u^*\|_{L^\infty(Q_T)}, \quad (t, x, y) \in Q_T. \]

In addition, if

\[ (\text{div } V_h^*)_{ij} + K_{i,j}^* \geq 0, \]

then we have

\[ 0 \leq \|u_h(t, x, y) - u^*\|_{L^\infty(Q_T)}, \quad (t, x, y) \in Q_T. \]

Obviously, since \( K \geq 0 \), (3.4) is satisfied if \( V_1 \) is nondecreasing in \( x \) and \( V_2 \) is nondecreasing in \( y \), or \( \text{div } V \) is uniformly positive by the definition of \( P_{Q_h} \) [21].

Define, for \( n = 0, \ldots, n_T \),

\[ \|u_h^n\|_{BV(\Omega)} = \sum_{i=0}^{n_x} \sum_{j=0}^{n_y} \left( |u_{i+1,j}^n - u_{i,j}^n| \Delta y_j + |u_{i,j+1}^n - u_{i,j}^n| \Delta x_i \right), \]

and set

\[ \Delta x_* = \min\{\Delta x_i, \ i = 1, \ldots, n_x\}, \quad \Delta y_* = \min\{\Delta y_j, \ j = 1, \ldots, n_y\}. \]

**Theorem 3.2 (total variation boundedness (TVB)).** Assume that (3.1) and for \( n = 0, \ldots, n_T - 1 \) the following CFL condition is satisfied:

\[ \Delta t^* \leq \frac{1}{D_{ij}^* + 3V_1^*/\Delta x_* + 3V_2^*/\Delta y_*}, \quad i = 1, \ldots, n_x, j = 1, \ldots, n_y. \]

Then, there is a constant \( C_1 \) depending solely on the data and \( T \) such that

\[ \|u_h\|_{L^\infty(J; BV(\Omega))} \leq C_1 \left\{ 1 + \|K\|_{L^\infty(J; BV(\Omega))} \right\} + D \left( \frac{1}{\Delta x_*} + \frac{1}{\Delta y_*} \right) \|u_D\|_{L^\infty(J; BV(\partial Q_0))}. \]

We remark that either in the case of \( D = 0 \) or in the case of \( u_D \) being constant in space, (3.7) shows that the total variation of the solution \( u_h \) is bounded. The latter case means that the total variation of the solution \( u_h \) in the one-dimensional case is always bounded since \( u_D \) is constant in this case. The numerical experiments given in section 8 show that the bounds in (3.7) and (3.8) below are sharp when \( D \neq 0 \) and \( \|u_D\|_{L^\infty(J; BV(\partial Q_0))} \neq 0 \), in the sense that the left-hand sides of the inequalities (3.7) and (3.8) blow up as \( \Delta x_* \) or \( \Delta y_* \) converge to zero (see Example 4 in section 8).
In the following, \( v_h \) stands for the approximate solution of (1.1) and (1.2) with the data \( v_{\text{init}} \) and \( v_T \) satisfying the conditions (3.1a), (3.1c), and (3.1d).

**Theorem 3.3 (Continuity with respect to data).** Assume that the hypotheses of Theorem 3.1 are satisfied for both sets of data. Then there exists a constant \( C_2 \) depending only on the data and \( T \) such that

\[
\| u_h - v_h \|_{L^\infty(J;L^1(\Omega))} \leq C_2 \left( 1 + D \left( \frac{1}{\Delta x_\ast} + \frac{1}{\Delta y_\ast} \right) \right) \times \| u_D - v_D \|_{L^\infty(J;L^1(\partial \Omega_2))} + \| u_{\text{init}} - v_{\text{init}} \|_{L^1(\Omega)}.
\]

As for the convergence result, we now consider a simple case where \( D = 0 \). In this case Theorem 3.2 implies the total variation boundedness of the numerical scheme as remarked above, which together with Theorem 3.3 yields the following convergence result (see section 7). For nonzero \( D \), concrete error estimates for the numerical scheme (2.2) will be obtained in the work mentioned earlier.

In the simple case where \( D = 0 \), the boundary conditions (1.2a) and (1.2b) are replaced by the following inflow boundary condition:

\[
u = u_D, \quad (x, y) \in \partial \Omega_-, \quad t \in J,
\]

where \( \partial \Omega_- = \{(x, y) \in \partial \Omega : (V \cdot n)(x, y) < 0\} \). We now extend the numerical flux introduced in (2.2) to the general setting

\[
f(u_{\text{left}}, u_{\text{right}}; \alpha) = u_{\text{left}} \alpha^+ + u_{\text{right}} \alpha^-.
\]

Also, we define

\[
C_0^1([0, T) \times \Omega) = \{ \varphi \in C^1(J \times \Omega) : \varphi(T, x, y) = 0, (x, y) \in \Omega \}.
\]

Then, a weak solution of the differential equation given by (1.1) with \( D = 0 \), (3.9), and (1.2c) is defined to be a function \( u \in L^\infty(J;BV(\Omega)) \) satisfying the weak formulation

\[
(u, \varphi_i)_{Q_T} + (uV \cdot \nabla \varphi)_{Q_T} + (u_{\text{init}}, \varphi)_{(t=0) \times \Omega} - (f(u, u_D; V \cdot n), \varphi)_{J \times (\partial \Omega)} - (K u, \varphi)_{Q_T} = 0 \quad \forall \varphi \in C_0^1([0, T) \times \Omega),
\]

where \((\cdot, \cdot)_S\) denotes the inner product in \( L^2(S) \) for some set \( S \). Note that the role of the flux \( f \) is to select the correct boundary value for \( u \) and that the smoothness hypothesis on \( V \) guarantees the uniqueness of weak solution to (3.10).

**Theorem 3.4 (convergence).** Assume that the hypotheses of Theorem 3.2 are satisfied. Then the sequence \( \{ u_h \}_{h>0} \) produced by the scheme (2.2) converges in \( L^\infty(J;L^1(\Omega)) \) to the unique solution of (3.10). Moreover, \( u \in L^\infty(J;BV(\Omega)) \).

**4. Proof of the maximum principle.** In this section we prove Theorem 3.1.

Let

\[
U^n = \max\{ u^n_{i,j}, 0 \leq i \leq n_x + 1, 0 \leq j \leq n_y + 1 \}.
\]

**Lemma 4.1.** Suppose that

\[
1 - \frac{\Delta t^n}{\Delta x_i} (V^{i+1/2,j} - V^{i-1/2,j}) - \frac{\Delta t^n}{\Delta y_j} (V^{i+1/2,j} - V^{i-1/2,j})
\]

\[
- \frac{D}{\Delta x_i} \left( \frac{1}{\Delta x_{i+1/2}} + \frac{1}{\Delta x_{i-1/2}} \right) - \frac{D}{\Delta y_j} \left( \frac{1}{\Delta y_{j+1/2}} + \frac{1}{\Delta y_{j-1/2}} \right) - K_{i,j}^n \Delta t^n \geq 0.
\]
Then, if
\begin{equation}
0 \leq u_{i,j}^n, \quad 0 \leq i \leq n_x + 1, \quad 0 \leq j \leq n_y + 1,
\end{equation}
we have for $0 \leq i \leq n_x + 1$ and $0 \leq j \leq n_y + 1$
\begin{equation}
0 \leq u_{i,j}^{n+1} \leq U^{n+1} \leq \left\{ 1 + \Delta t^n \max_{I_i^x \times I_j^y} \{|\text{div} \, V^n_h|\} \right\} U^n.
\end{equation}
In addition, if
\begin{equation}
(\text{div} \, V^n_h)_{i,j} + K^n_{i,j} \geq 0,
\end{equation}
we have
\begin{equation}
0 \leq u_{i,j}^{n+1} \leq U^n, \quad 0 \leq i \leq n_x + 1, \quad 0 \leq j \leq n_y + 1.
\end{equation}

Proof. For $i = 1, \ldots, n_x$ and $j = 1, \ldots, n_y$, it follows from (2.2) that
\begin{equation}
u_{i,j}^{n+1} = A^n_{i+1,j} u^n_{i+1,j} + A^n_{i,j+1} u^n_{i,j+1} + B^n_{i,j} u^n_{i,j} + E^n_{i-1,j} u^n_{i-1,j} + E^n_{i,j-1} u^n_{i,j-1},
\end{equation}
where
\begin{align*}
A^n_{i+1,j} &= -\frac{\Delta t^n}{\Delta x_i} V^n_{1,i+1/2,j} + \frac{D \Delta t^n}{\Delta x_i \Delta x_{i+1/2}}, \\
A^n_{i,j+1} &= -\frac{\Delta t^n}{\Delta y_j} V^n_{2,i,j+1/2} + \frac{D \Delta t^n}{\Delta y_j \Delta y_{j+1/2}}, \\
B^n_{i,j} &= 1 - \frac{\Delta t^n}{\Delta x_i} (V^n_{2,i,j+1/2} - V^n_{2,i,j-1/2}) - \frac{\Delta t^n}{\Delta y_j} (V^n_{2,i,j+1/2} - V^n_{2,i,j-1/2}) \\
&\quad - \frac{D \Delta t^n}{\Delta x_i} \left( \frac{1}{\Delta x_{i+1/2}} + \frac{1}{\Delta x_{i-1/2}} \right) - \frac{D \Delta t^n}{\Delta y_j} \left( \frac{1}{\Delta y_{j+1/2}} + \frac{1}{\Delta y_{j-1/2}} \right) - K^n_{i,j} \Delta t^n, \\
E^n_{i-1,j} &= \frac{\Delta t^n}{\Delta x_i} V^n_{1,i-1/2,j} + \frac{D \Delta t^n}{\Delta x_i \Delta x_{i-1/2}}, \\
E^n_{i,j-1} &= \frac{\Delta t^n}{\Delta y_j} V^n_{2,i,j-1/2} + \frac{D \Delta t^n}{\Delta y_j \Delta y_{j-1/2}}.
\end{align*}
Then, by (4.1), we see that
\begin{equation}
A^n_{i+1,j}, A^n_{i,j+1}, B^n_{i,j}, E^n_{i-1,j}, E^n_{i,j-1} \geq 0,
\end{equation}
so that, by (4.2),
\begin{equation}u_{i,j}^{n+1} \geq 0, \quad i = 1, \ldots, n_x, \quad j = 1, \ldots, n_y.
\end{equation}
Furthermore, by the definition of $Q_h$ and (4.2),
\begin{equation}u_{i,j}^{n+1} \leq \left( 1 - \frac{\Delta t^n}{\Delta x_i} (V^n_{1,i+1/2,j} - V^n_{1,i-1/2,j}) \\
- \frac{\Delta t^n}{\Delta y_j} (V^n_{2,i,j+1/2} - V^n_{2,i,j-1/2}) - K^n_{i,j} \Delta t^n \right) U^n \\
= (1 - \Delta t^n (\text{div} \, V^n_h)_{i,j} - K^n_{i,j} \Delta t^n) U^n,
\end{equation}
which implies (4.3) immediately since $K \geq 0$, and together with (4.4) this yields (4.5).

LEMMA 4.2. If for $i = 1, \ldots, n_x$ and $j = 1, \ldots, n_y$

\( u_{i,j}^{n+1} \leq e^{t^{n+1}V_2(U^n)} \), \( i = 1, \ldots, n_x, j = 1, \ldots, n_y \).

Consequently, by (3.1a), (3.3) follows. Then iterating (4.3) on the assumption (3.1a). Let the results be true up to $n_D$ and (2.3b) yield

\[ \Delta t^n \leq \frac{1}{D_{ij}^* + 2\|V_1^n\|_{L^\infty(\Omega)}/\Delta x_i + 2\|V_2^n\|_{L^\infty(\Omega)}/\Delta y_j}, \]

where $D_{ij}^*$ is defined as in Theorem 3.1, then (4.1) is satisfied.

The lemma follows obviously from the inequality (4.1) and the definition of $D_{ij}^*$.

We are now ready to prove Theorem 3.1 by means of induction on $n$.

**Proof of Theorem 3.1.** For $n = 0$, the results (3.3) and (3.5) follow trivially from the assumption (3.1a). Let the results be true up to $n$. By Lemma 4.2 and (2.3a), (3.2) clearly implies (4.1). Then iterating (4.3) on $n$ and using the induction hypothesis and (2.3b) yield

\[ 0 \leq u_{i,j}^{n+1} \leq e^{t^{n+1}V_2(U^n)} \), \( i = 1, \ldots, n_x, j = 1, \ldots, n_y \).

Consequently, by (3.1a), (3.3) follows.

If (3.4) is true, so is (4.4). Then, in this case, it follows from (4.5) and the induction hypothesis that

\[ 0 \leq u_{i,j}^{n+1} \leq U^n \), \( i = 1, \ldots, n_x, j = 1, \ldots, n_y \),

which implies (3.5) by (3.1a).

**5. Proof of total variation boundedness.** In this section we prove Theorem 3.2. In order to fix ideas, let

\[ \partial \Omega_2 = \{(x, y) : x = 0, 0 < y < 1\} \cup \{(x, y) : x = 1, 0 < y < 1\}, \]

\[ \partial \Omega_1 = \partial \Omega \setminus \partial \Omega_2; \]

other cases can be treated similarly.

LEMMA 5.1. For $i = 0$ and $j = 1, \ldots, n_y$

\[ u_{0,j}^{n+1} - u_{0,j}^n = u_{0,j}^n - u_{0,j}^{n+1} + \left( -\frac{\Delta t^n}{\Delta x_1} V_{0,1,i+1/2}^n + \frac{D \Delta t^n}{\Delta x_3/2 \Delta x_1} (u_{0,j}^n - u_{0,j}^{n+1}) \right) \]

\[ + \left( \frac{\Delta t^n}{\Delta y_j} V_{2,1,j+1/2}^n + \frac{D \Delta t^n}{\Delta y_j \Delta y_{j+1/2} \Delta y_{j-1/2}} (u_{1,j+1}^n - u_{0,j+1}^n) \right) \]

\[ + \left( 1 - \frac{\Delta t^n}{\Delta x_1} V_{1,1,j+1/2}^n + \frac{\Delta t^n}{\Delta y_j} V_{2,1,j+1/2}^n - \frac{\Delta t^n}{\Delta y_j} V_{2,1,j-1/2}^n \right) \]

\[ - \frac{D \Delta t^n}{\Delta x_0 \Delta x_1} - \frac{D \Delta t^n}{\Delta y_j \Delta y_{j+1/2} \Delta y_{j-1/2}} \right) (u_{1,j}^n - u_{0,j}^n) \]

\[ + \left( \frac{\Delta t^n}{\Delta y_j} (V_{2,1,j+1/2}^n + \frac{D \Delta t^n}{\Delta y_j \Delta y_{j+1/2} \Delta y_{j-1/2}}) (u_{1,j+1}^n - u_{0,j}^n) \right) \]

\[ - \left( \frac{\Delta t^n}{\Delta x_1} (V_{1,1,j+1/2}^n - V_{1,1,j-1/2}^n) + \frac{\Delta t^n}{\Delta y_j} (V_{2,1,j+1/2}^n - V_{2,1,j-1/2}^n) \right) u_{0,j}^n \]

\[ + \left( \frac{\Delta t^n}{\Delta y_j} (V_{2,1,j+1/2}^n + \frac{D \Delta t^n}{\Delta y_j \Delta y_{j+1/2} \Delta y_{j-1/2}}) (u_{0,j+1}^n - u_{0,j}^n) \right) \]

\[ - \left( \frac{\Delta t^n}{\Delta y_j} V_{2,1,j+1/2}^n + \frac{D \Delta t^n}{\Delta y_j \Delta y_{j+1/2} \Delta y_{j-1/2}} \right) (u_{0,j}^n - u_{0,j-1}^n) - K_{1,j}^n \Delta t^n u_{1,j}^n. \]
for \( i = 1, \ldots, n_x - 1 \) and \( j = 1, \ldots, n_y \)

\[
\begin{align*}
    u_{i+1,j}^{n+1} - u_{i,j}^{n+1} &= \left( -\frac{\Delta t^n}{\Delta x_{i+1}} V^{n-1}_{1,i+3/2,j} + \frac{D \Delta t^n}{\Delta x_{i+1} \Delta x_{i+3/2}} \right) (u_{i+2,j}^{n} - u_{i+1,j}^{n}) \\
    &\quad + \left( -\frac{\Delta t^n}{\Delta y_{j}} V^{n-1}_{2,i+1,j+1/2} + \frac{D \Delta t^n}{\Delta y_{j} \Delta y_{j+1/2}} \right) (u_{i+1,j+1}^{n} - u_{i,j+1}^{n}) \\
    &\quad + \left\{ 1 - \frac{\Delta t^n}{\Delta x_{i+1}} (V^{n}_{1,i+3/2,j} - V^{n}_{1,i+1/2,j}) - \frac{\Delta t^n}{\Delta y_{j}} (V^{n}_{2,i+1,j+1/2} - V^{n}_{2,i+1,j-1/2}) \\
    &\quad - V^{n}_{2,i+1,j-1/2} \right\} + \frac{\Delta t^n}{\Delta x_{i}} V^{n-1}_{1,i+1/2,j} - \frac{\Delta t^n}{\Delta x_{i+1}} V^{n+1}_{1,i+1/2,j} \\
    &\quad + \frac{\Delta t^n}{\Delta y_{j}} (V^{n-1}_{2,i+1,j+1/2} - V^{n+1}_{2,i+1,j-1/2}) - \frac{D \Delta t^n}{\Delta x_{i+1} \Delta x_{i+1/2}} \left( \frac{1}{\Delta x_{i}} + \frac{1}{\Delta x_{i+1}} \right) \\
    &\quad - \frac{D \Delta t^n}{\Delta y_{j}} \left( \frac{1}{\Delta y_{j+1/2}} + \frac{1}{\Delta y_{j-1/2}} \right) - K^{n}_{i+1,j+1} \Delta t^n \right) (u_{i+1,j}^{n} - u_{i,j}^{n}) \\
    &\quad + \left( \frac{\Delta t^n}{\Delta x_{i}} V^{n+1}_{1,i-1/2,j} + \frac{D \Delta t^n}{\Delta x_{i} \Delta x_{i-1/2}} \right) (u_{i-1,j}^{n} - u_{i,j}^{n}) \\
    &\quad + \left( \frac{\Delta t^n}{\Delta y_{j}} V^{n+1}_{2,i+1,j-1/2} + \frac{D \Delta t^n}{\Delta y_{j} \Delta y_{j-1/2}} \right) (u_{i+1,j-1}^{n} - u_{i,j-1}^{n}) \\
    &\quad - \frac{\Delta t^n}{\Delta y_{j}} (V^{n-1}_{2,i+1,j+1/2} - V^{n-1}_{2,i+1,j-1/2}) (u_{i,j+1}^{n} - u_{i,j}^{n}) \\
    &\quad + \frac{\Delta t^n}{\Delta y_{j}} (V^{n+1}_{2,i+1,j-1/2} - V^{n-1}_{2,i+1,j-1/2}) (u_{i,j-1}^{n} - u_{i,j}^{n}) \\
    &\quad + \Delta t^n ((\text{div } V^{n}_{h})_{i+1,j} - (\text{div } V^{n}_{h})_{i,j}) u_{i,j}^{n} \\
    &\quad + \Delta t^n (K^{n}_{i,j} - K^{n}_{i+1,j}) u_{i,j}^{n} .
\end{align*}
\]

and for \( i = n_x \) and \( j = 1, \ldots, n_y \)

\[
\begin{align*}
    u_{n_x+1,j}^{n+1} - u_{n_x,j}^{n+1} &= u_{n_x+1,j}^{n+1} - u_{n_x,j}^{n+1} \\
    &\quad + \left\{ 1 + \frac{\Delta t^n}{\Delta x_{n_x}} V^{n-1}_{1,n_x+1/2,j} + \frac{\Delta t^n}{\Delta y_{j}} V^{n-1}_{2,n_x,j+1/2} - \frac{\Delta t^n}{\Delta y_{j}} V^{n+1}_{2,n_x,j-1/2} \\
    &\quad - \frac{D \Delta t^n}{\Delta x_{n_x} \Delta x_{n_x+1/2}} - \frac{D \Delta t^n}{\Delta y_{j} \Delta y_{j+1/2}} \right\} (u_{n_x+1,j}^{n} - u_{n_x,j}^{n}) \\
    &\quad + \left( \frac{\Delta t^n}{\Delta x_{n_x}} V^{n+1}_{1,n_x-1/2,j} + \frac{D \Delta t^n}{\Delta x_{n_x} \Delta x_{n_x-1/2}} \right) (u_{n_x,j}^{n} - u_{n_x,j-1}^{n}) \\
    &\quad + \left( \frac{\Delta t^n}{\Delta y_{j}} V^{n+1}_{2,n_x,j+1/2} + \frac{D \Delta t^n}{\Delta y_{j} \Delta y_{j+1/2}} \right) (u_{n_x+1,j+1}^{n} - u_{n_x,j+1}^{n}) \\
    &\quad + \left( \frac{\Delta t^n}{\Delta y_{j}} V^{n+1}_{2,n_x,j-1/2} + \frac{D \Delta t^n}{\Delta y_{j} \Delta y_{j-1/2}} \right) (u_{n_x+1,j-1}^{n} - u_{n_x,j-1}^{n}) \\
    &\quad + \frac{\Delta t^n}{\Delta x_{n_x}} (V^{n}_{1,n_x+1/2,j} - V^{n}_{1,n_x-1/2,j}) \\
    &\quad + \frac{\Delta t^n}{\Delta y_{j}} (V^{n}_{1,n_x,j+1/2} - V^{n}_{1,n_x,j-1/2}) \right) u_{n_x,j}^{n} .
\end{align*}
\]
Then, the proof is completed by simple algebraic manipulations on $u_{i,j}^{n+1} - u_{i,j}^n$.

Proof. From (2.2), we see that
\[
\begin{align*}
u_{i,j}^{n+1} &= \left( -\frac{\Delta t^n}{\Delta x_i} V_{1,i+1/2,j}^{n-1} + \frac{D \Delta t^n}{\Delta x_i \Delta x_{i+1/2}} \right) (u_{i+1,j}^n - u_{i,j}^n) \\
&\quad + \left( -\frac{\Delta t^n}{\Delta y_j} V_{2,i,j+1/2}^{n-1} + \frac{D \Delta t^n}{\Delta y_j \Delta y_{j+1/2}} \right) (u_{i,j+1}^n - u_{i,j}^n) \\
&\quad + \left( -\frac{\Delta t^n}{\Delta x_i} V_{1,i-1/2,j}^{n} - \frac{D \Delta t^n}{\Delta x_i \Delta x_{i-1/2}} \right) (u_{i-1,j}^n - u_{i,j}^n) \\
&\quad + \left( \frac{\Delta t^n}{\Delta y_j} V_{2,i,j-1/2}^{n} - \frac{D \Delta t^n}{\Delta y_j \Delta y_{j-1/2}} \right) (u_{i,j-1}^n - u_{i,j}^n) - K_n \Delta t^n u_{i,j}^n \\
&\quad + \left( 1 - \frac{\Delta t^n}{\Delta y_j} (V_{1,i+1/2,j}^{n} - V_{1,i-1/2,j}^{n}) - \frac{\Delta t^n}{\Delta x_i} (V_{2,i,j+1/2}^{n} - V_{2,i,j-1/2}^{n}) \right) u_{i,j}^n.
\end{align*}
\]

Then, the proof is completed by simple algebraic manipulations on $u_{i+1,j}^n - u_{i,j}^n$. \hfill \Box

Lemma 5.2. Assume that
\begin{equation}
\label{eq:51a}
0 \leq u_{i,j}^n, \quad i = 0, \ldots, n_x + 1, j = 0, \ldots, n_y + 1,
\end{equation}
and for $i = 1, \ldots, n_x$ and $j = 1, \ldots, n_y$
\begin{equation}
\label{eq:51b}
\Delta t^n \leq \frac{1}{D_{t,i}^n + 3 ||V_{h,i}^n||_{L^\infty(\Omega)}/\Delta x_i + 3 ||V_{h,j}^n||_{L^\infty(\Omega)}/\Delta y_j}.
\end{equation}

Then there is a constant $C_3 = C_3(C_0)$ such that
\[
\begin{align*}
||u_{h}^{n+1}||_{BV(\Omega)} &\leq (1 + C_3 \Delta t^n \|[\nabla V^n]\|_{L^\infty(\Omega)}) ||u_{h}^{n}||_{BV(\Omega)} + \Delta t^n \|[\nabla V^n]\|_{BV(\Omega)} U^n \\
&\quad + \Delta t^n C_3 (||V_{h}^n||_{L^\infty(\Omega)} + K^n) U^n + \Delta t^n ||K^n||_{BV(\Omega)} U^n \\
&\quad + 2 \Delta t^n \sum_{j=0}^{n_y} \left( ||V_{h,j}^n||_{L^\infty(\Omega)} + \frac{D}{\Delta y_{j+1/2}} \right) |u_{0,j}^{n+1} - u_{0,j}^n| \\
&\quad + 2 \Delta t^n \sum_{j=0}^{n_y} \left( ||V_{h,j}^n||_{L^\infty(\Omega)} + \frac{D}{\Delta y_{j+1/2}} \right) |u_{n_x,j+1}^n - u_{n_x,j}^n| \\
&\quad + \sum_{j=0}^{n_y} |u_{0,j}^{n+1} - u_{0,j}^n| \Delta y_j + \sum_{j=0}^{n_x} |u_{n_x+1,j}^{n+1} - u_{n_x+1,j}^n| \Delta y_j.
\end{align*}
\]

Proof. From (5.1b) we see that the coefficients of the terms between the brackets \{\} in the expressions of Lemma 5.1 are nonnegative. Then the estimate of a typical term is given as follows:
\[
|u_{i+1,j}^{n+1} - u_{i,j}^{n+1}| \leq \left( -\frac{\Delta t^n}{\Delta x_{i+1}} V_{1,i+3/2,j}^{n-1} + \frac{D \Delta t^n}{\Delta x_{i+1} \Delta x_{i+3/2}} \right) |u_{i+2,j}^{n+1} - u_{i+1,j}^n|
\]
Thus simple algebraic manipulations and use of (2.3c) yield the desired result. □

Proof of Theorem 3.2. Note that the CFL condition (3.6) implies (5.1b) by (2.3a). Then the result (3.7) follows by iterating on \( n \) the inequality in Lemma 5.2 and using Theorem 3.1 and (2.3b). □

6. Proof of continuity with respect to data. In this section we prove Theorem 3.3 and a result on equicontinuity in time of the approximate solution, Proposition 6.4 below. We recall that \( u_h \) stands for the solution of (2.2) with the data \( v_D \) and \( v_{init} \).

Lemma 6.1. For \( n = 0, \ldots, n_T \), \( i = 1, \ldots, n_x \), and \( j = 1, \ldots, n_y \) we have

\[
\begin{align*}
u_{i,j}^{n+1} - u_{i,j}^n &= A_{i+1,j}^n(u_{i+1,j}^n - u_{i+1,j}^n) + A_{i,j+1}^n(u_{i,j+1}^n - u_{i,j+1}^n) \\
&+ B_{i,j}^n(u_{i,j}^n - v_{i,j}^n) + E_{i,j}^n(u_{i,j}^n - v_{i,j}^n) \\
&+ E_{i,j}^{n-1}(u_{i,j}^n - v_{i,j}^n),
\end{align*}
\]

where \( A_{i+1,j}^n \), \( A_{i,j+1}^n \), \( B_{i,j}^n \), \( E_{i,j}^n \), and \( E_{i,j}^{n-1} \) are defined as in the proof of Lemma 4.1.

The result easily follows from (2.2).

Lemma 6.2. Suppose that (4.1) is satisfied. Then

\[
||u_h^{n+1} - v_h^{n+1}||_{L^1(\Omega)} \leq \Delta t^n \left(||V_h^1||_{L^\infty(\Omega)} + ||V_h^2||_{L^\infty(\Omega)} + D \left( \frac{1}{\Delta x_*} + \frac{1}{\Delta y_*} \right) \right) \\
\times ||u_D - v_D||_{L^1(\partial \Omega_2)} + ||u_h^n - v_h^n||_{L^1(\Omega)}.
\]

Proof. Since, by (4.1), the coefficients in the equality of Lemma 6.1 are nonnegative, we see that
Then the lemma follows by multiplying this inequality by $\Delta x_i\Delta y_j$, adding over $i$, $j$, and rearranging terms imply the desired result. \hfill \Box

Now, Theorem 3.3 can be easily seen from Lemma 6.2.

**Lemma 6.3.** Assume that the CFL condition (4.1) is satisfied. Then,

$$
||u_h^n - u_h^0||_{L^1(\Omega)} \leq 2\Delta t^0 \left( ||V_{h1}^0||_{L^\infty(\Omega)} + ||V_{h2}^0||_{L^\infty(\Omega)} + D \left( \frac{1}{\Delta x_*} + \frac{1}{\Delta y_*} \right) ||u_0^{\text{init}}||_{BV(\Omega)} + \Delta t^0 \left( ||V_{h1}^0||_{BV(\Omega)} + ||V_{h2}^0||_{BV(\Omega)} + K^* \right) u^* \right).
$$

**Proof.** By (2.2), we observe that

$$
|u_{i,j}^1 - u_{i,j}^0| \leq \left( -\Delta t^0 \frac{\Delta V_{1,i+1/2,j}}{\Delta x_i} + \Delta t^0 \frac{D}{\Delta x_*} \right) |u_{i+1,j}^0 - u_{i,j}^0| + \left( -\Delta t^0 \frac{\Delta V_{2,i+1/2,j}}{\Delta y_j} + \Delta t^0 \frac{D}{\Delta y_*} \right) |u_{i,j+1}^0 - u_{i,j}^0| + \left( \Delta t^0 \frac{\Delta V_{1,i-1/2,j}}{\Delta x_i} + \Delta t^0 \frac{D}{\Delta x_*} \right) |u_{i-1,j}^0 - u_{i,j}^0| + \left( \Delta t^0 \frac{\Delta V_{2,i-1/2,j}}{\Delta y_j} + \Delta t^0 \frac{D}{\Delta y_*} \right) |u_{i,j-1}^0 - u_{i,j}^0| + \Delta t^0 \left( \frac{1}{\Delta x_i} (V_{1,i+1/2,j}^0 - V_{1,i-1/2,j}^0) + \frac{1}{\Delta y_j} (V_{2,i+1/2,j}^0 - V_{2,i-1/2,j}^0) \right) u_{i,j}^0.
$$

Then the lemma follows by multiplying this inequality by $\Delta x_i\Delta y_j$ and adding the resulting one over $i$, $j$. \hfill \Box

**Proposition 6.4 (equicontinuity in time).** Under the assumptions of Theorem 3.1, there is a constant $C_4$ depending only on the data and $T$ such that for $n = 0, \ldots, n_T$,

$$
||u_{h}^{n+1} - u_{h}^n||_{L^1(\Omega)} \leq C_4 \Delta t \left( 1 + D \left( \frac{1}{\Delta x_*} + \frac{1}{\Delta y_*} \right) \right) \times \left( ||u_0^{\text{init}}||_{BV(\Omega)} + ||u_D||_{L^1(\partial\Omega^2; BV(J))} \right).
$$

**Proof.** We take $v_{h}^{n+1} = u_{h}^n$ in Lemma 6.2 and use Lemma 6.3 to obtain the result. \hfill \Box

**7. A convergence analysis.** In this section we prove Theorem 3.4 by applying the ideas used in [10] for analyzing the one-dimensional drift-diffusion semiconductor device equations. We point out that the analysis here is much simpler than that given in [10]. The reason is that here we are using the standard entropy $| \cdot |$, while a smoother entropy has been used there, which requires much work to estimate the distance between the smooth entropy and the standard one. We also emphasize the difference between the present analysis and that used in classical conservation laws;
in the present case the delicate part is how to handle the boundary terms in the “entropy form” Θ (see (7.3) below), while an unbounded domain is treated in the classical conservation laws.

It should be emphasized that this whole section concerns the case of $D = 0$ and that, although the differential equation (1.1) is linear, techniques which have been originally developed for nonlinear hyperbolic conservation laws will be used.

The proof of Theorem 3.4 proceeds as follows. First, we prove that there is a subsequence $\{u_{h'}\}_{h' > 0}$ converging to a limit $u$. Then, we show that

$$\lim_{h' \to 0} R(u_{h'}, \varphi) = R(u, \varphi),$$

(7.1a)

$$R(u, \varphi) = 0,$$

(7.1b)

for $\varphi \in C^1_0([0, T) \times \Omega)$, where $R(\cdot, \cdot)$ defines the left-hand side of (3.10). Since the weak solution of (3.10) is assumed unique, this completes the proof of Theorem 3.4.

As in classical conservation laws, (7.1) follows from the following result [13]:

$$\lim_{h' \to 0} \Theta(u_{h'}, c; V_{h'}; \varphi) = \Theta(u, c; V; \varphi) \quad \forall c \in \mathbb{R}, \varphi \in C^1(\overline{Q}_T),$$

(7.2a)

$$\Theta(u, c; V; \varphi) \leq 0 \quad \forall c \in \mathbb{R}, 0 \leq \varphi \in C^1(\overline{Q}_T),$$

(7.2b)

where $\Theta$ is defined in (7.3) below. Most of this section is devoted to proving this result.

7.1. The entropy form. The entropy form $\Theta(u, c; V; \varphi)$ with boundary terms included is defined as follows:

$$\Theta(u, c; V; \varphi) = - ([u - c]_T \varphi)_{Q_T} - ([u - c] \nabla \varphi)_{Q_T}$$

$$+ ([u - c]_T \varphi)_{\{t = T\} \times \Omega} - ([u_{\text{left}} - c]_T \varphi)_{\{t = 0\} \times \Omega}$$

$$+ (G(u - c, u_D - c; V \cdot \nu, \varphi))|_{\partial \Omega}$$

$$- (H(u, c)(\text{div } V + K), \varphi)_{Q_T} + (K[u - c], \varphi)_{Q_T},$$

(7.3)

where $c \in \mathbb{R}$, $\varphi \in C^1(\overline{Q}_T)$, and the “entropy flux” $G$ and the function $H$ are defined by

$$G(u_{\text{left}}, u_{\text{right}}; V \cdot \nu) = |u_{\text{left}}|(V \cdot \nu)^+ + |u_{\text{right}}|(V \cdot \nu)^-,$$

$$H(u, c) = |u - c| - u \text{ sign}(u - c).$$

The motivation of the form $\Theta$ can be given as in the one-dimensional case [10].

7.2. A convergent subsequence. In this subsection we prove the existence of a convergent subsequence $\{u_{h'}\}_{h' > 0}$. 

**Lemma 7.1.** Assume that the hypotheses of Theorem 3.2 are satisfied. Then there exists a subsequence $\{u_{h'}\}_{h' > 0}$ converging in $L^\infty(J; L^1(\Omega))$ to a limit $u$ in $L^\infty(J; BV(\Omega)) \cap C^0(J; L^1(\Omega))$.

**Proof.** We note that the ideas in [13] can be used to prove the lemma. In [13], a discrete version of Azcoli–Arzelà theorem was used. In the present case with $D = 0$, the equicontinuity in time is provided by Proposition 6.4, and the compactness of the range is given by Theorem 3.2. Also, the regularity result on $u$ follows from the convergence and Theorem 3.2. ☐
7.3. Proof of (7.2a). Here we prove (7.2a) under a condition.

**Lemma 7.2.** Suppose that for \( c \in \mathbb{R} \) and nonnegative \( \varphi \in C^1(\overline{Q_T}) \)

\[
\lim_{h' \to 0} \Theta(u_{h'}, c; V_{h'}; \varphi) \leq 0.
\]

Then

\[
\lim_{h' \to 0} \Theta(u_{h'}, c; V_{h'}; \varphi) = \Theta(u, c; V; \varphi).
\]

**Proof.** First, for every nonnegative \( \varphi \in C^1_0([0, T) \times \Omega) \), (7.5) follows from Lemma 7.1 and the standard argument in the classical conservation laws [13]. Also, since \( u \in C^0(J; L^1(\Omega)) \) by Lemma 7.1, the same result holds for \( \varphi \in C^1_0([0, T] \times \Omega) \).

We now consider the case where \( \varphi \in C^1_0(J \times (0, 1) \times (0, 1)) \). Since we are mainly concerned with the boundary term associated with the edge \( \{ x = 0, 0 < y < 1 \} \), it suffices to consider \( \varphi(t, x, y) \) of this form \( \omega(t, y)\xi(x) \). Then, set

\[
g_{h'}(x) = \int_0^T \int_0^1 |u_{h'} - c| |V_{h'}(t, y)| dy dt,
\]

\[
g_{h'}(0-) = \int_0^T \int_0^1 \{ |u_{D, \Delta t} - c| |V_{h'}(t, 0, y)| + |u_{h'}(t, 0+, y) - c| |V_{h'}^-(t, 0, y)| \} \omega dt dy,
\]

and rewrite \( \Theta(u_{h'}, c; V_{h'}; \varphi) \) as follows:

\[
\Theta(u_{h'}, c; \varphi) = - \left( |u_{h'} - c|, \xi \frac{\partial \omega}{\partial t} \right)_{Q_T} - \left( |u_{h'} - c| |V_{h'}|, \frac{\partial \omega}{\partial y} \right)_{Q_T} \]

\[
- \int_0^1 g_{h'}(x)\xi'(x) dx - g_{h'}(0-\)\xi(0)
\]

\[-(H(u_{h'}, c)(\text{div } V + K), \xi\omega)_{Q_T} + (K|u_{h'} - c|, \xi\omega)_{Q_T}.
\]

Since the sequence \( \{ u_{h'}(\cdot, 0+, \cdot) \}_{h' > 0} \) is bounded in \( L^\infty(J \times (0, 1)) \) by Theorem 3.1, there is a subsequence \( \{ u_{h''} (\cdot, 0+, \cdot) \}_{h'' > 0} \) converging in \( L^\infty(J \times (0, 1)) \)—weak* to a limit \( \bar{u} \). Let \( \gamma_{\bar{u}, \omega} \) be the Young measure corresponding to \( \bar{u} \). Then, by Lemma 7.1 and (2.3a), we see that

\[
\lim_{h'' \to 0} \Theta(u_{h''}, c; V_{h''}; \varphi) = - \left( |u - c|, \xi \frac{\partial \omega}{\partial t} \right)_{Q_T} - \left( |u - c| |V_2|, \frac{\partial \omega}{\partial y} \right)_{Q_T} \]

\[
- \int_0^1 g(x)\xi'(x) dx - g_0\xi(0)
\]

\[-(H(u, c)(\text{div } V + K), \xi\omega)_{Q_T} + (K|u - c|, \xi\omega)_{Q_T},
\]

where

\[
g(x) = \int_0^T \int_0^1 |u - c| |V_1 \omega(t, y)| dy dt,
\]

\[
g_0 = \int_0^T \int_0^1 \{ |u_{D} - c| |V_1^+(t, 0, y)| + w(t, y) |V_1^-(t, 0, y)| \} \omega dt dy,
\]

\[
w(t, y) = \int_0^{u^*} |\lambda - c| d\gamma_{\bar{u}, \omega}(\lambda),
\]

\[
u^{**} = e^{TV_0^*} u^*.
\]
Thus, to prove (7.5), it suffices to prove that \( g_0 = g^* \), where

\[
(7.9) \quad g^* = \int_0^T \int_0^1 \left\{ |u_D - c| V_1^+(t, 0, y) + |u(t, 0+, y) - c| V_1^-(t, 0, y) \right\} \omega dt dy.
\]

Take \( \xi \) such that its support is contained in \([0, \epsilon] \). Then, by (7.4) and Theorem 3.1, it follows from (7.6) and (7.7) that

\[
- \int_0^1 g(x) \xi'(x) dx - g_0 \xi(0) \leq C \epsilon \|\xi\|_{L^1(0,1)}.
\]

Since \( \epsilon \) is arbitrary, this inequality yields

\[
(7.10) \quad g(0+) - g_0 \leq 0.
\]

Choose \( c \in \mathbb{R} \) such that \( |u - c| = \alpha(u - c) \) for some \( \alpha \in \mathbb{R} \). Then, by (7.8) and the definition of \( \gamma_{t,y} \),

\[
w(t, y) = \alpha(\tilde{u} - c),
\]

so that, by (7.7), (7.10) becomes

\[
\alpha \left\{ \int_0^T \int_0^1 (u(t, 0+, y) - u_D) V_1^+(t, 0, y) \omega dy dt + (u(t, 0+, y) - \tilde{u}) V_1^-(t, 0, y) \omega dy dt \right\} \leq 0.
\]

Since the sign of \( \alpha \) is arbitrary and this inequality is true for any nonnegative \( \omega \in C_0^1(T \times (0,1)) \), we have

\[
(7.11a) \quad V_1^+(t, 0, y)(u(t, 0+, y) - u_D) = 0 \quad \text{almost everywhere (a.e.) in } J \times (0,1),
\]

\[
(7.11b) \quad V_1^-(t, 0, y)(u(t, 0+, y) - \tilde{u}) = 0 \quad \text{a.e. in } J \times (0,1).
\]

Finally, by (7.7)–(7.9) and (7.11), we see that

\[
g(0+) - g_0 = g^* - g_0
\]

\[
= \int_0^T \int_0^1 (|u(t, 0+, y) - c| - w(t, y)) V_1^-(t, 0, y) \omega dy dt
\]

\[
= \int_0^T \int_0^1 (|\tilde{u}(t, y) - c| - w(t, y)) V_1^-(t, 0, y) \omega dy dt
\]

\[
\geq \int_0^T \int_0^1 \left( \int_0^u (\lambda - c) d\gamma_{t,y}(\lambda) - w(t, y) \right) V_1^-(t, 0, y) \omega dy dt
\]

\[
= 0,
\]

which together with (7.10) implies that \( g_0 = g^* \). This completes the proof of the case where \( \varphi \in C_0^1(J \times [0,1] \times (0,1)) \). The same argument applies to the remaining three cases.

It is now clear that it suffices to prove (7.4). This is done in the next two subsections.
7.4. A discrete entropy inequality. The following discrete entropy inequality will be needed for obtaining an upper bound for $\Theta(u_h', c; V_{h'}; \varphi)$.

**Lemma 7.3.** Under the CFL condition (3.2), we have for $c \in \mathbb{R}$

\[
|u_{i,j}^{n+1} - c| - (1 - K_{i,j}^n \Delta t^n)|u_{i,j}^n - c| + \frac{\Delta t^n}{\Delta x_i} (G_{i+1/2,j}^n - G_{i-1/2,j}^n) + \frac{\Delta t^n}{\Delta y_j} (G_{i,j+1/2}^n - G_{i,j-1/2}^n) - H(u_{i,j}^{n+1}, c) \{ (\text{div } V_{i,j}^n) + K_{i,j}^n \} \Delta t^n \leq 0,
\]

where

\[
G_{i+1/2,j}^n = V_{i+1/2,j}^n |u_{i,j}^n - c| + V_{i+1/2,j}^n |u_{i+1,j}^n - c|, \\
G_{i,j+1/2}^n = V_{i,j+1/2}^n |u_{i,j}^n - c| + V_{i,j+1/2}^n |u_{i,j+1}^n - c|.
\]

**Proof.** From (2.2) and the definition of the mixed finite element space $Q_h$, we have for $c \in \mathbb{R}$

\[
u_{i,j}^{n+1} - c = \left( -\frac{\Delta t^n}{\Delta x_i} V_{i+1/2,j}^n \right) (u_{i,j+1}^n - c) + \left( -\frac{\Delta t^n}{\Delta y_j} V_{2,i,j+1/2}^n \right) (u_{i,j+1}^n - c) + \left( 1 - \frac{\Delta t^n}{\Delta x_i} (V_{i+1/2,j}^n - V_{i,j+1/2}^n) - \frac{\Delta t^n}{\Delta y_j} (V_{2,i,j+1/2}^n - V_{2,i,j-1/2}^n) - K_{i,j}^n \right) \Delta t^n (u_{i,j}^n - c) + \left( \frac{\Delta t^n}{\Delta x_i} V_{i,j-1/2}^n \right) (u_{i,j}^{n-1} - c) + \left( \frac{\Delta t^n}{\Delta y_j} V_{2,i,j-1/2}^n \right) (u_{i,j}^n - c) - \Delta t^n [(\text{div } V_{i,j}^n) + K_{i,j}^n] c.
\]

Note that the term between the brackets is nonnegative by (3.2). Thus, the lemma follows by multiplying this expression by sign $(u_{i,j}^{n+1} - c)$. \(\Box\)

7.5. An upper bound of entropy form. In this section we obtain an upper bound for $\Theta(u_h', c; V_{h'}; \varphi)$, which implies the inequality (7.4). We first have the following decomposition of $\Theta(u_h', c; V_{h'}; \varphi)$:

For $\varphi \in C^1(Q_T)$ let

\[
\varphi_{i,j}^n = \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \varphi(t^n, x, y) \, dx \, dy,
\]

\[
\varphi_{i,j+1/2}^n = \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \varphi(t^n, x, y) \, dt \, dx \, dy,
\]

\[
\varphi_{i+1/2,j}^n = \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \varphi(t, x_{i+1/2}, y) \, dt \, dy,
\]

\[
\varphi_{i,j+1/2}^n = \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \varphi(t, x, y_{j+1/2}) \, dt \, dx.
\]

**Lemma 7.4 (decomposition of $\Theta$).** We have

\[
\Theta(u_h', c; V_{h'}; \varphi) = \Theta_{\text{ent}}(u_h', c; V_{h'}; \varphi) + \Theta_{\text{com}}(u_h', c; V_{h'}; \varphi).
\]
where (with arguments omitted)

\[
\Theta_{\text{cnt}} = \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \left\{ |u^{n+1}_{i,j} - c| - (1 - K_{i,j}^n \Delta t^n)|u^n_{i,j} - c| \\
+ \frac{\Delta t^n}{\Delta x_i}(G_{i+1/2,j}^n - G_{i-1/2,j}^n) + \frac{\Delta t^n}{\Delta y_j}(G_{i,j+1/2}^n - G_{i,j-1/2}^n) \\
- H(u^{n+1}_{i,j}, c) \left( (\text{div} V_h^n)_{i,j} + K_{i,j}^n \Delta t^n \right) \phi_{i,j}^{n+1} \Delta x_i \Delta y_j, \right\}
\]

and

\[
\Theta_{\text{com}} = \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \left\{ (|u^n_{i+1,j} - c| - |u^n_{i,j} - c|)(-V^n_{1,i+1/2,j})(\phi_{i,j}^{n+1} - \phi_{i-1/2,j}^{n+1/2}) \\
+ (|u^n_{i-1,j} - c| - |u^n_{i,j} - c|)(V^n_{1,i-1/2,j})(\phi_{i,j}^{n+1} - \phi_{i-1/2,j}^{n+1/2}) \right\} \Delta t^n \Delta y_j \\
+ \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \left\{ (|u^n_{i,j+1} - c| - |u^n_{i,j} - c|)(-V^n_{2,i,j+1/2})(\phi_{i,j}^{n+1} - \phi_{i,j-1/2,j}^{n+1/2}) \\
+ (|u^n_{i,j-1} - c| - |u^n_{i,j} - c|)(V^n_{2,i,j-1/2})(\phi_{i,j}^{n+1} - \phi_{i,j-1/2,j}^{n+1/2}) \right\} \Delta t^n \Delta x_i \\
- \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} (\text{div} V_h^n)_{i,j} u^n_{i,j} \text{sign}(u^n_{i,j} - c)(\phi_{i,j}^{n+1} - \phi_{i,j}^{n+1/2}) \Delta t^n \Delta x_i \Delta y_j \\
+ \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} K_{i,j}^n H(u^{n+1}_{i,j}, c)(\phi_{i,j}^{n+1} - \phi_{i,j}^{n+1/2}) \Delta t^n \Delta x_i \Delta y_j \\
+ \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} K_{i,j}^n |u^n_{i,j} - c|(|\phi_{i,j}^n - \phi_{i,j}^{n+1}|) \Delta t^n \Delta x_i \Delta y_j \\
+ \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} (\text{div} V_h^n)_{i,j} (H(u^{n+1}_{i,j}, c) - H(u^n_{i,j}, c))\phi_{i,j}^{n+1} \Delta t^n \Delta x_i \Delta y_j \\
+ \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} K_{i,j}^n (H(u^{n+1}_{i,j}, c) - H(u^n_{i,j}, c))\phi_{i,j}^{n+1/2} \Delta t^n \Delta x_i \Delta y_j. 
\]

**Proof.** From the definition of \( \Theta \) and the fact that \( \text{div} V_h \) is piecewise constant, we have

\[
\Theta = \Psi_t + \Psi_x + \Psi,
\]

where

\[
\Psi_t = - \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} |u^n_{i,j} - c|(\phi_{i,j}^{n+1} - \phi_{i,j}^{n+1/2}) \Delta x_i \Delta y_j + \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} |u^{n_T}_{i,j} - c|\phi_{i,j}^{n_T} \Delta x_i \Delta y_j \\
- \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} |u^0_{i,j} - c|\phi_{i,j}^0 \Delta x_i \Delta y_j,
\]

\( || \) denotes the absolute value.
\[\Psi_x = - \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} [u_{i,j}^n - c(V_{i,j}^{n+1/2} - V_{i,j-1/2}^n)] \Delta t^n \Delta y_j \]

\[\Psi = - \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} (\text{div} V_h)_{i,j} u_{i,j}^n \text{sign} (u_{i,j}^n - c) \varphi_{i,j}^{n+1/2} \Delta t^n \Delta x_i \Delta y_j \]

\[\Psi = - \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} K_{i,j}^n H(u_{i,j}^n, c) \varphi_{i,j}^n \Delta t^n \Delta x_i \Delta y_j \]

\[+ \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} K_{i,j}^n |u_{i,j}^n - c| \varphi_{i,j}^n \Delta t^n \Delta x_i \Delta y_j.\]

Then, simple algebraic manipulations yield the desired result.

**Lemma 7.5 (upper bound of \(\Theta\)).** Suppose that the conditions of Theorem 3.2 are satisfied. Then, there is a constant \(C_5\) depending solely on the data and \(T\) such that for any \(\varphi \in C^1(\Omega)\), \(\varphi \geq 0\)

\[\Theta_{\text{ext}} \leq 0, \]

\[\Theta_{\text{com}} \leq C_5(1 + |c|) \{ |\Delta x| \| \varphi_x \|_{L^\infty(\Omega_T)} + |\Delta y| \| \varphi_y \|_{L^\infty(\Omega_T)} + |\Delta t| \| \varphi \|_{L^\infty(\Omega_T)} \} + \| \Sigma_2 \|_{L^2(\Omega_T)}.\]

**Proof.** The first inequality follows immediately from Lemmas 7.3 and 7.4. Also, observe that

\[|\varphi_{i,j}^{n+1} - \varphi_{i,j}^{n+1/2}| \leq \frac{1}{2} \Delta x_i \| \varphi_x \|_{L^\infty(J \times \Omega)} + \frac{1}{2} \Delta t^n \| \varphi_t \|_{L^\infty(J \times \Omega)},\]

\[|\varphi_{i,j}^{n+1} - \varphi_{i,j+1/2}^{n+1}| \leq \frac{1}{2} \Delta y_j \| \varphi_y \|_{L^\infty(J \times \Omega)} + \frac{1}{2} \Delta t^n \| \varphi_t \|_{L^\infty(J \times \Omega)},\]

\[|\varphi_{i,j}^{n+1} - \varphi_{i,j}^{n+1/2}| \leq \frac{1}{2} \Delta t^n \| \varphi_t \|_{L^\infty(J \times \Omega)},\]

\[|\varphi_{i,j}^{n+1} - \varphi_{i,j}^{n}| \leq \Delta t^n \| \varphi_t \|_{L^\infty(J \times \Omega)}.\]

Then, if an integration by parts on \(n\) is applied to the last two terms in the expression of \(\Theta_{\text{com}}\), the second inequality follows from Theorems 3.1 and 3.2, Proposition 6.4, Lemma 7.4, and (2.3a).

We are now in a position to prove Theorem 3.4.

**Proof of Theorem 3.4.** From Lemma 7.1 there exists a subsequence \(\{u_h\}_{h>0}\) converging in \(L^\infty(J; L^1(\Omega))\) to a limit \(u\). Now, by Lemma 7.5, we have

\[\lim_{h' \to 0} \Theta(u_{h'}, c; V_{h'}; \varphi) \leq 0.\]
for every $c \in \mathbb{R}$ and nonnegative $\varphi \in C^1(\overline{Q_T})$. Thus, by Lemma 7.2, we see that

$$\lim_{h' \to 0} \Theta(u_{h'}, c; V_{h'}; \varphi) = \Theta(u, c; V; \varphi) \leq 0.$$ 

This implies that $u$ is the unique solution of (3.10). Consequently, the whole sequence $\{u_h\}_{h > 0}$ converges to $u$, and, thus, Theorem 3.4 is proven. \qed

8. Numerical results. This section reports on numerical results with the finite element method (2.2) for three problems. They are designed to show the performance of the method and to indicate the convergence properties. In all examples the CFL condition (3.6) is required to hold.

**Example 1.** In this example we consider a convecting Gaussian hill in one space dimension. Specifically, we solve (1.1) with $\Phi = 1$, $V = 10$, $D = 0.1$, and $K = 0$ on the interval $[0, 6]$. The initial datum $u_{\text{init}}$ is given by

$$u_{\text{init}}(x) = e^{-\pi x^2}.$$ 

As a pure initial value problem, this leads to the analytical solution

$$u_a(t, x) = \frac{1}{\sqrt{1 + 4\pi Dt}} e^{-\frac{\pi}{1+4\pi Dt} x^2}.$$ 

We obtain an initial boundary value problem with the same solution by imposing the Dirichlet boundary condition $u(t, 0) = u_{a}(t, 0)$, $u(t, 6) = u_{a}(t, 6)$.

In Figure 1 we display the analytical solution $u_a$ and the approximate solution $u_h$ at time $T = 0.25$. In Table 1 we display the errors and their respective orders of convergence at the same time. From the table we see that the scheme is first-order accurate both in $L^1$ and in $L^\infty$ for the concentration. This shows that the scheme (2.2) is first-order accurate in both spaces when the solution of the differential equation is smooth. Also, Figure 1 agrees with the stability property given in Theorem 3.1. Finally, our numerical experiments (not shown here) report that if the CFL condition (3.2) is violated, then the stability result (3.5) and the TVB (3.7) are no longer valid.

**Example 2.** In this example we consider a problem whose solution displays a discontinuity. The problem has the data $\Phi = 1$, $V = -0.5$, and $K = 0$. The boundary and initial conditions are given by

$$u(t, 0) = u_a(t, 0), \quad u(t, 6) = u_a(t, 6).$$ 

The exact and approximate “nonviscous” solution (i.e., in the case of $D = 0$) and the “viscous” solution with $D = 10^{-3}$ at $T = 0.5$ are displayed in Figure 2. Notice that the biggest error in the approximation of $u$ occurs around the location of the discontinuity $x = 0.75$. In Table 2 we show the errors and their convergence orders with $D = 0$. Note that the orders of convergence in $L^1$ and $L^\infty$ are nearly one-half. This implies that the presence of discontinuity has an effect on the convergence. Finally, from Figure 2, we see that the “nonviscous” solution is quite close to the “viscous” solution.

**Example 3.** In the third example we consider a two-dimensional problem which has a shock. The data are set as follows: $\Omega = (0, 1)^2$, $\Phi = 1$, $V = (\cos(\frac{\pi}{2}), \sin(\frac{\pi}{2}))$, 

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The "—", "- - -", and "···" represent the exact $u$ and approximate solution $u_h$ with $h = .001$ and $h = .01$.

**TABLE 1**
Convergence of $u_h$ in $(0, 6)$ at $T = .25$.

<table>
<thead>
<tr>
<th>1/Δx</th>
<th>$L^\infty$-error (×10^2)</th>
<th>$L^\infty$-order</th>
<th>$L^1$-error (×10^2)</th>
<th>$L^1$-order</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>6.09</td>
<td>-</td>
<td>5.10</td>
<td>-</td>
</tr>
<tr>
<td>100</td>
<td>3.57</td>
<td>0.76</td>
<td>2.50</td>
<td>1.03</td>
</tr>
<tr>
<td>200</td>
<td>1.89</td>
<td>0.91</td>
<td>1.20</td>
<td>0.99</td>
</tr>
<tr>
<td>400</td>
<td>0.99</td>
<td>0.93</td>
<td>0.62</td>
<td>1.02</td>
</tr>
<tr>
<td>800</td>
<td>0.51</td>
<td>0.98</td>
<td>0.29</td>
<td>1.09</td>
</tr>
</tbody>
</table>

$K = 0$, and $D = 10^{-3}$. The Neumann and Dirichlet boundaries $\partial \Omega_1$ and $\partial \Omega_2$ are defined by

$$
\partial \Omega_1 = \{(x, y) : 0 \leq x \leq 1, y = 1\},
\partial \Omega_2 = \partial \Omega \setminus \partial \Omega_1,
$$

and the boundary and initial data by
The approximate solution of this problem obtained using the method (2.2) with $\Delta x = \Delta y = 10^{-2}$ at time $T = 2$ is shown in Figure 3. The graph clearly shows that the method can capture the shock around the location $y = 1/2$. 

**Table 2**

*Convergence of $u_h$ in $(0, 1)$ at $T = .5$.*

<table>
<thead>
<tr>
<th>$1/\Delta x$</th>
<th>$L^\infty$-error ($\times 10^2$)</th>
<th>$L^\infty$-order</th>
<th>$L^1$-error ($\times 10^2$)</th>
<th>$L^1$-order</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>8.65</td>
<td>-</td>
<td>9.75</td>
<td>-</td>
</tr>
<tr>
<td>100</td>
<td>6.15</td>
<td>0.48</td>
<td>8.09</td>
<td>0.26</td>
</tr>
<tr>
<td>200</td>
<td>5.02</td>
<td>0.29</td>
<td>6.32</td>
<td>0.37</td>
</tr>
<tr>
<td>400</td>
<td>3.76</td>
<td>0.43</td>
<td>4.36</td>
<td>0.54</td>
</tr>
<tr>
<td>800</td>
<td>2.49</td>
<td>0.60</td>
<td>2.77</td>
<td>0.65</td>
</tr>
</tbody>
</table>

$u_D = \begin{cases} 
1, & x = 0, \frac{1}{2} < y < 1, \\
0, & \text{elsewhere}, 
\end{cases}$

$u_{\text{init}} = \begin{cases} 
1, & 0 \leq x \leq 1, \frac{1}{2} < y \leq 1, \\
0, & \text{elsewhere}. 
\end{cases}$
The approximate solution $u_h$ on $(0, 1)^2$.

**Table 3**

TVB bounds of $u_h$ in $\Omega$ at $T = 1$.

<table>
<thead>
<tr>
<th>$1/\Delta x$</th>
<th>10</th>
<th>20</th>
<th>40</th>
<th>80</th>
<th>160</th>
<th>320</th>
</tr>
</thead>
<tbody>
<tr>
<td>TVB</td>
<td>4.8935</td>
<td>9.5977</td>
<td>20.451</td>
<td>43.507</td>
<td>93.602</td>
<td>198.30</td>
</tr>
</tbody>
</table>

**Example 4.** In the final example we test the sharpness of the bounds appearing in (3.7) and (3.8) when $D \neq 0$ and $\|u_D\|_{L^\infty(J, BV(\partial \Omega_2))} \neq 0$. The same set of data are chosen as in Example 3 except that the initial and boundary data are determined by

$u_{\text{init}}(x, y) = x, \quad (x, y) \in \Omega$,

$u_D(x, y) = x, \quad (x, y) \in \partial \Omega_2$,

where

$\partial \Omega_2 = \{(x, y) : y = 1, 0.2 < x < 0.4\} \cup \{(x, y) : y = 1, 0.6 < x < 0.8\}$,

$\partial \Omega_1 = \partial \Omega \setminus \partial \Omega_2$.

Uniform partitions of $\Omega$ into rectangles are exploited. The TVB bounds on different meshes at $T = 1$ are given in Table 3. From this table we see that the left-hand side of the inequality (3.7) blows up as $h = \Delta x = \Delta y$ converges to zero. Similar results are observed for the bound in (3.8) (not shown here).

**9. A concluding remark.** A new finite element method for numerically solving the two-dimensional convection-dominated transport equation in ground water has been formulated and analyzed in this paper. The primary computational advantage
of the method is that it is local and, thus, fully parallelizable, and it is conservative. The stability properties of this method and its convergence in a suitable topology have been established. Moreover, the numerical results have shown that the method is first-order accurate when the solution is smooth and is one-half-order accurate when the solution has discontinuities, and that the method is nonoscillatory and shock-capturing. Future work will be devoted to obtaining error estimates for both cases of zero and nonzero coefficient $D$.

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**REFERENCES**


