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Analysis of the pressure projection stabilization method for the Darcy and coupled Darcy–Stokes flows

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Abstract In this paper, we systematically analyze the pressure projection stabilization method for the Darcy and coupled Darcy–Stokes flow problems in multiple dimensions. Stability results for this stabilization method are established. For the Darcy flow, optimal error estimates in the divergence norm for velocity and suboptimal error estimates in the L^2-norm for pressure are obtained, and a superconvergence result for the pressure is derived; a local postprocessing scheme is constructed to generate optimal error estimates in the L^2-norm for pressure. For the coupled Darcy–Stokes flow, error estimates of optimal order are obtained in terms of the energy norm of velocity and pressure. Numerical results are presented to check the theory developed.

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1 Introduction

The mixed finite element method has been a popular method for solving the partial differential equations arising in solid and fluid mechanics [8, 14, 16]. Its popularity is due to the fact that in some cases, a vector variable (e.g., a fluid velocity) is the primary variable in which one is interested. Then, the mixed method is developed to approximate both this variable and a scalar variable (e.g., a pressure) simultaneously and to give accurate approximations of both variables. The mixed finite element formulation uses two different approximate spaces. These two spaces must be chosen carefully so they satisfy an *inf–sup* stability condition for the mixed method to be stable. There exist rich choices for these special spaces for the equations of solid and fluid mechanics [8, 14, 16].

Much attention has recently been attracted to using the equal-order finite element pairs (e.g., $P_1 - P_1$, the linear function pair and $Q_1 - Q_1$, the bilinear function pair) for the fluid mechanics equations, particularly for the Stokes and Navier–Stokes equations [3, 4, 9, 17, 24, 25, 30]. While they do not satisfy the inf–sup stability condition, these element pairs offer simple and practical uniform data structure and adequate accuracy. Many stabilization techniques have been proposed to stabilize these element pairs such as penalty, pressure projection, and residual stabilization methods [18, 20, 26, 30]. Among these methods, the pressure projection stabilization method is a preferable choice in that it is free of stabilization parameters, does not require any calculation of high-order derivatives or edge-based data structures, and can be implemented at the element level [4, 17, 24, 25]. As formulated in [4, 24, 25, 29], it is based on two local Gauss integrals. Recent studies have been focused on stability and convergence of stabilization of the lowest equal-order finite element pair $P_1 - P_1$ or $Q_1 - Q_1$ using this type of stabilization for the Stokes and Navier–Stokes equations [4, 24, 25].

There have been attempts to use the pressure projection stabilization method for solving the second-order partial differential equations modeling the Darcy flow. Numerical results were reported using low-order finite element pairs [5, 11], and stability and convergence studies were given in [11]. However, a complete analysis is still lacking for the second-order equations. The analysis of the mixed finite element method for these equations is much more delicate than for the Stokes equations since the latter equations are naturally given in mixed form and different sets of the Sobolev spaces are used; the pair $H^1(\Omega) \times L^2(\Omega)$ is used for the Stokes equations while the set $H(\text{div}; \Omega) \times L^2(\Omega)$ is employed for the second-order equations. In this paper, we will provide a systematical analysis for the pressure projection stabilization method for the second-order elliptic Darcy flow problems in multiple dimensions. The analysis will focus on a superconvergence result and a local postprocessing scheme. Toward that end, stability results for the finite element solution are considered, and optimal error estimates in the divergence norm for velocity and suboptimal error estimates in the L^2-norm for pressure are obtained. These results are similar to those obtained in [11]. The local postprocessing scheme is used to generate optimal error estimates in the L^2-norm for pressure. Numerical results are presented to check the theory developed.

To see how the current approach and analysis can be generalized to other problems, a coupled Darcy–Stokes flow problem is considered. This coupled problem arises in many engineering applications, such as in porous media flow where the Darcy flow applies in the ordinary porous part and the Stokes flow applies in the vuggy (cavity) part of a porous medium [1, 15]. The development of approximation methods for solving the coupled Darcy–Stokes flow problems has attracted much interest [19, 22, 27]. In this paper, we generalize the pressure projection stabilization method for the Darcy flow to a coupled Darcy–Stokes flow problem, and error estimates of optimal order are obtained in terms of the energy norm of velocity and pressure. Preliminary computational results were presented in [31].

The rest of this paper is organized as follows: In the next section, the basic notation, the differential equation of the Darcy flow, and its mixed finite element method are stated. Then, in the third section, a stability result is shown. Basic error estimates are derived in the fourth section, and a superconvergence result is proved in the fifth section. A local postprocessing scheme is studied in the sixth section. The extension to the coupled Darcy–Stokes flow problem is carried out in the seventh to ninth sections. Numerical experiments are presented in the tenth section. Finally, concluding remarks are given in the 11th section. The theory will be presented for general pairs of mixed finite element spaces V_h and W_h. Their prototypes are the spaces of piecewise polynomials of degree k, $k \geq 1$.

2 Darcy flow

Let Ω denote a polygonal domain in $\Re^d$ ($d = 2$ or 3), $f \in L^2(\Omega)$, and $K = (K_{ij})$ be a $d \times d$ matrix-valued function on Ω. We assume that there exist constants $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \|\xi\|^2 \leq \sum_{i,j=1}^{d} K_{ij}(x)\xi_i\xi_j \leq \alpha_2\|\xi\|^2 \quad \forall x \in \Omega,$$
$$\xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \Re^d. \quad (2.1)$$

Consider the Darcy flow problem:

$$-\text{div}(K\nabla p) = f \quad \text{in } \Omega, \quad (2.2a)$$

$$p = 0 \quad \text{on } \partial\Omega. \quad (2.2b)$$

It is well known that problems (2.2a) and (2.2b) have a unique solution. In the subsequent analysis, we implicitly assume that p has, at each step, the regularity required by the context. The exact requirements are easily obtained from inspection of the arguments. Note that if Ω is convex and $f \in H^s(\Omega)$ for some $s > 0$, then $p \in H^r(\Omega)$ for some number $r > 2$ which depends on s and Ω.

Standard definitions are used for the Sobolev spaces $W^{m,r}(\Omega)$, with the norm $\|\cdot\|_{m,r}$ and the seminorm $|\cdot|_{m,r}$, $m, r \geq 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$. The notation $(\cdot,\cdot)$ indicates the inner product on the domain Ω.

In order to state a mixed formulation for (2.2a) and (2.2b), we define the space

$$V = H(\text{div}; \Omega) = \{v \in (L^2(\Omega))^d : \text{div } v \in L^2(\Omega)\}, \quad (2.3)$$

with the norm

$$\|v\|^2_{H(\text{div};\Omega)} = \sum_{i=1}^{d} \|v_i\|_0^2 + \|\text{div } v\|_0^2, \quad (2.4)$$

where $v = (v_1, v_2, \ldots, v_d)$. Also, we will use the space, with the standard L^2-norm,

$$W = L^2(\Omega).$$

Setting

$$u = -K\nabla p, \tag{2.5}$$

(2.2a) becomes

$$\operatorname{div} u = f. \tag{2.6}$$

For the Darcy flow, p and u are the fluid pressure and velocity, respectively; (2.5) and (2.6) represent Darcy's law and continuity equation, respectively [15].

Multiply (2.5) by $v \in V$ and integrate over Ω to see that

$$(K^{-1}u, v) = -(v, \nabla p).$$

Applying Green's formula to the right-hand side of this equation gives

$$(K^{-1}u, v) = (\operatorname{div} v, p),$$

where we used the boundary condition (2.2b). Also, multiplying (2.6) by any $w \in W$, we see that

$$(\operatorname{div} u, w) = (f, w).$$

Then, we have a system for u and p:

$$(K^{-1}u, v) - (\operatorname{div} v, p) = 0 \quad \forall v \in V, \tag{2.7a}$$

$$(\operatorname{div} u, w) = (f, w) \quad \forall w \in W. \tag{2.7b}$$

This is the mixed variational form of (2.2a) and (2.2b). If u and p satisfy (2.5) and (2.6), they also satisfy (2.7a) and (2.7b). The converse also holds if p is sufficiently smooth (e.g., if $p \in H^2(\Omega)$). Furthermore, systems (2.7a) and (2.7b) have a unique solution $u \in V$ and $p \in W$.

Introduce the bilinear form

$$B((u, p), (v, w)) = (K^{-1}u, v) - (\operatorname{div} v, p) + (\operatorname{div} u, w), \quad (u, p),\ (v, w) \in V \times W.$$

Systems (2.7a) and (2.7b) are written as

$$B((u, p), (v, w)) = (f, w) \quad \forall (v, w) \in V \times W. \tag{2.8}$$

Let us consider now a decomposition T_h of Ω into convex subdomains $\{T\}$, with which we associate the spaces $V_h \subset V$ and $W_h \subset W$.

The mixed finite element method for (2.2a) and (2.2b) is defined: Find $u_h \in V_h$ and $p_h \in W_h$ such that

$$(K^{-1}u_h, v) - (\operatorname{div} v, p_h) = 0 \quad \forall v \in V_h, \tag{2.9a}$$

$$(\operatorname{div} u_h, w) = (f, w) \quad \forall w \in W_h. \tag{2.9b}$$

In general, an arbitrary choice of the approximate spaces $V_h \subset V$ and $W_h \subset W$ does not satisfy the inf–sup condition uniformly in h:

$$\sup_{v \in V_h} \frac{(\operatorname{div} v, w)}{\|v\|_{H(\operatorname{div};\Omega)}} \geq \beta \|w\|_0 \qquad \forall w \in W_h,$$

where the constant $\beta > 0$ is independent of h. Thus, we assume that another approximate space $\bar{W}_h \subset W$ exists such that

$$\bar{W}_h \subset W_h, \tag{2.10a}$$

$$V_h \times \bar{W}_h \text{ satisfies the } inf\!-\!sup \text{ condition uniformly in } h \tag{2.10b}$$

$$\operatorname{div} V_h \subset W_h. \tag{2.10c}$$

Let $\bar{P}_h : L^2(\Omega) \to \bar{W}_h$ be the standard L^2-projection, which satisfies

$$\|\bar{P}_h p\|_0 \leq C\|p\|_0, \quad p \in L^2(\Omega). \tag{2.11}$$

The central idea of the mixed finite element method developed is to use the spaces of equal-order mixed finite element pairs $V_h \times W_h$ that do not satisfy the inf–sup stability condition. To overcome this deficiency, we introduce the pressure projection stabilization term [4, 24, 25]

$$G_h(p, q) = (p - \bar{P}_h p, q - \bar{P}_h q). \tag{2.12}$$

Now, the pressure projection stabilization method reads: Find $u_h \in V_h$ and $p_h \in W_h$ such that

$$(K^{-1}u_h, v) - (\operatorname{div} v, p_h) = 0 \quad \forall v \in V_h, \tag{2.13a}$$

$$(\operatorname{div} u_h, w) + G_h(p_h, w) = (f, w) \quad \forall w \in W_h. \tag{2.13b}$$

Introduce the bilinear form

$$B_h((u_h, p_h), (v, w)) = (K^{-1}u_h, v) - (\operatorname{div} v, p_h) + (\operatorname{div} u_h, w) + G_h(p_h, w),\ (u_h, p_h),\ (v, w) \in V_h \times W_h.$$

Systems (2.13a) and (2.13b) are written as

$$B_h((u_h, p_h), (v, w)) = (f, w) \quad \forall (v, w) \in V_h \times W_h. \tag{2.14}$$

We end this section with an example for the spaces V_h, W_h, and $\bar{W}_h$.

Example Let T_h be a shape-regular triangulation of the polygonal domain Ω into a union of triangles or tetrahedra [14, 16]. Associated with T_h, we define

$$\begin{aligned} V_h &= \{v \in V : v|_T \in (P_k(T))^d,\ T \in T_h\}, \\ W_h &= \{w \in W : w|_T \in P_k(T),\ T \in T_h\}, \\ \bar{W}_h &= \{w \in W : w|_T \in P_{k-1}(T),\ T \in T_h\}, \end{aligned} \tag{2.15}$$

where $P_k(T)$ represents the space of polynomials of degree not greater than k on set T, $k \geq 1$. These spaces satisfy assumption (2.10a), (2.10b), and (2.10c) in both two ($d = 2$) and three ($d = 3$) dimensions [6, 7]. Note that the inf–sup stable pair for the velocity and pressure (u, p) is the Brezzi–Douglas–Marini space ($d = 2$) or the Brezzi–Douglas–Durán–Fortin space $V_h \times \bar{W}_h$ ($d = 3$).

3 Stability

To show a stability result, we define the norm on $V \times W$

$$\|(v, w)\| = \left(\|v\|^2_{H(\mathrm{div};\Omega)} + \|w\|_0^2\right)^{1/2}, \quad (v, w) \in V \times W.$$

It follows from assumption (2.10b) that there is a linear operator $\Pi_h : (H^1(\Omega))^d \to V_h$ such that the boundedness and orthogonality relation hold [8, 14]:

$$\|\Pi_h v\|_{H(\mathrm{div};\Omega)} \leq C\|v\|_{H(\mathrm{div};\Omega)}, \quad v \in (H^1(\Omega))^d,\ d = 2 \text{ or } 3, \tag{3.1}$$

and

$$(\mathrm{div}(v - \Pi_h v), w) = 0, \quad v \in (H^1(\Omega))^d, \quad w \in \bar{W}_h. \tag{3.2}$$

Let $P_h : L^2(\Omega) \to W_h$ be the standard L^2-projection. It satisfies

$$\|P_h w\|_0 \leq C\|w\|_0, \quad w \in L^2(\Omega). \tag{3.3}$$

The following stability result is similar to a stability result shown in [11]. For completeness, we give a proof.

Theorem 3.1 *Under assumptions (2.10a), (2.10b), and (2.10c), there are positive constants C and β, independent of h, such that*

$$B_h((u_h, p_h), (v_h, w_h)) \leq C\|(u_h, p_h)\| \, \|(v_h, w_h)\|, \quad (u_h, p_h), (v_h, w_h) \in V_h \times W_h, \tag{3.4}$$

and

$$\sup_{(v_h, w_h) \in V_h \times W_h} \frac{B_h((u_h, p_h), (v_h, w_h))}{\|(v_h, w_h)\|} \geq \beta\|(u_h, p_h)\|, \quad (u_h, p_h) \in V_h \times W_h. \tag{3.5}$$

Proof The continuity property (3.4) can easily be shown using the definition of the bilinear form $B_h(\cdot, \cdot)$ and the norm $\|\cdot\|$. It suffices to prove the weak coercivity (3.5).

Given $p_h \in W_h$, define $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ by

$$\begin{aligned} -\Delta\phi &= p_h \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Let $v = -\nabla\phi$ so $\mathrm{div}\, v = p_h$. Consequently, we see that

$$\|v\|_{H(\mathrm{div};\Omega)} \leq C\|p_h\|_0. \tag{3.6}$$

For $u_h \in V_h$, define $v_h = u_h - \epsilon_1 \Pi_h v \in V_h$ and $w_h = p_h + \epsilon_2 \mathrm{div}\, u_h$, where the constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ are yet to be determined. It can be seen from (3.1) and (3.6) that

$$\|(v_h, w_h)\| \leq C\|(u_h, p_h)\|. \tag{3.7}$$

Next, it follows from the definition of the bilinear form $B_h(\cdot, \cdot)$ that

$$\begin{aligned} B_h((u_h, p_h), (v_h, w_h)) = &(K^{-1}u_h, u_h) - \epsilon_1(K^{-1}u_h, \Pi_h v) \\ &+ \epsilon_2\|\mathrm{div}\, u_h\|_0^2 + \epsilon_1(\mathrm{div}\, \Pi_h v, p_h) \\ &+ G_h(p_h, p_h) + \epsilon_2 G_h(p_h, \mathrm{div}\, u_h). \end{aligned} \tag{3.8}$$

Using (3.1) and (3.6), we see that

$$|\epsilon_1(K^{-1}u_h, \Pi_h v)| \leq \frac{1}{2}(K^{-1}u_h, u_h) + \epsilon_1^2 C_1\|p_h\|_0^2. \tag{3.9}$$

Also, by (2.11), we have

$$|\epsilon_2 G_h(p_h, \mathrm{div}\, u_h)| \leq \frac{1}{2}G_h(p_h, p_h) + \epsilon_2^2 C_2\|\mathrm{div}\, u_h\|_0^2. \tag{3.10}$$

Note that, by (3.2),

$$\begin{aligned} \|p_h\|_0^2 &= (p_h, \mathrm{div}\, v) \\ &= (p_h - \bar{P}_h p_h, \mathrm{div}\, v) + (\bar{P}_h p_h, \mathrm{div}\, v) \\ &= (p_h - \bar{P}_h p_h, \mathrm{div}\, v) + (\bar{P}_h p_h, \mathrm{div}\, \Pi_h v) \\ &= (p_h - \bar{P}_h p_h, \mathrm{div}(v - \Pi_h v)) + (p_h, \mathrm{div}\, \Pi_h v), \end{aligned}$$

and, by (3.1) and (3.6),

$$\begin{aligned} |(p_h - \bar{P}_h p_h, \mathrm{div}(v - \Pi_h v))| &\leq \|p_h - \bar{P}_h p_h\|_0 \|\mathrm{div}(v - \Pi_h v)\|_0 \\ &\leq C G_h^{1/2}(p_h, p_h)\|p_h\|_0 \\ &\leq \frac{1}{2}\|p_h\|_0^2 + C G_h(p_h, p_h). \end{aligned}$$

Consequently, we obtain

$$\frac{1}{2}\|p_h\|_0^2 \leq C_3 G_h(p_h, p_h) + (\mathrm{div}\Pi_h v, p_h). \tag{3.11}$$

Combing (3.8)–(3.11) gives

$$\begin{aligned} B_h((u_h, p_h), (v_h, w_h)) \geq &\frac{1}{2}(K^{-1}u_h, u_h) + \epsilon_2(1 - \epsilon_2 C_2)\|\mathrm{div}\, u_h\|_0^2 \\ &+ \frac{\epsilon_1}{2}(1 - 2\epsilon_1 C_1)\|p_h\|_0^2 + \frac{1}{2}(1 - 2\epsilon_1 C_3)G_h(p_h, p_h). \end{aligned}$$

Choosing

$$\epsilon_1 = \max\left(\frac{1}{4C_1}, \frac{1}{4C_3}\right), \quad \epsilon_2 = \frac{1}{2C_2},$$

we see that

$$B_h((u_h, p_h), (v_h, w_h)) \geq C\|(u_h, p_h)\|^2. \tag{3.12}$$

Finally, combining (3.7) and (3.12) yields (3.5). □

It follows from Theorem 3.1 that system (2.14) has a unique solution, which satisfies

$$\|(u_h, p_h)\| \leq C\|f\|_0. \tag{3.13}$$

Example (continued) The example given in (2.15) for the finite element spaces V_h, W_h, and $\bar{W}_h$ satisfies conditions (3.1) and (3.2) [6, 7].

4 Error estimates

In this section, we derive basic error estimates for the mixed finite element solution $u_h \in V_h$ and $p_h \in W_h$. Subtracting (2.14) from (2.8) results in the error equation

$$B((u, p) - (u_h, p_h), (v_h, w_h)) = G_h(p_h, w_h), \quad (v_h, w_h) \in V_h \times W_h. \tag{4.1}$$

The following best possible error estimate holds. A similar convergence result was stated in [11]; for completeness, we give a complete proof.

Theorem 4.1 *Under assumptions* (2.10a), (2.10b)*, and* (2.10c)*, it holds that*

$$\|(u - u_h, p - p_h)\| \le C \inf_{(v_h, w_h) \in V_h \times \bar{W}_h} \|(u - v_h, p - w_h)\|, \tag{4.2}$$

where (u, p) *and* (u_h, p_h) *are the respective solutions of* (2.8) *and* (2.14).

Proof For any $(v_h, w_h) \in V_h \times \bar{W}_h$, set

$$\xi_h = u_h - v_h, \quad \eta_h = p_h - w_h.$$

It follows from (3.5) that

$$\beta \|(\xi_h, \eta_h)\| \le \sup_{(x_h, y_h) \in V_h \times W_h} \frac{B_h((\xi_h, \eta_h), (x_h, y_h))}{\|(x_h, y_h)\|}. \tag{4.3}$$

Since $G_h(w_h, y_h) = 0$ for $w_h \in \bar{W}_h$, inequality (4.3), together with (4.1) and the continuity of the bilinear form $B(\cdot, \cdot)$, implies

$$\beta \|(\xi_h, \eta_h)\| \le \sup_{(x_h, y_h) \in V_h \times W_h} \frac{B((u - v_h, p - w_h), (x_h, y_h))}{\|(x_h, y_h)\|} \le C \|(u - v_h, p - w_h)\|.$$

As a result, (4.2) comes from this inequality and the triangle inequality. □

Example (continued) With the example given in (2.15), the estimate (4.2) gives

$$\|u - u_h\|_0 + \|\mathrm{div}(u - u_h)\|_0 + \|p - p_h\|_0 \le Ch^k (\|u\|_k + \|\mathrm{div}\, u\|_k + \|p\|_k), \tag{4.4}$$

which is optimal for u in the divergence norm and suboptimal for u and p in the L^2-norm in terms of the convergence rate. As the numerical experiments will show in the seventh section, these are the best estimates one can obtain with method (2.14). The reason is that the approximation property of the lower-order space $\bar{W}_h$ always pollutes the global accuracy. Note that the error estimate (4.4) is consistent with that produced by the residual stabilized mixed method and the least-squares method [21, 26].

Improvements on convergence rates will be made using a local postprocessing technique considered in the sixth section.

5 Superconvergence

In this section, we consider the case where $\mathrm{div}\, V_h = \bar{W}_h$ (a subset $\mathrm{div}\, V_h \subset \bar{W}_h$ suffices). Also, the approximation property holds for the linear operators Π_h and $\bar{P}_h$:

$$\|v - \Pi_h v\|_0 \le Ch\|v\|_1, \quad \|w - \bar{P}_h w\|_0 \le Ch\|w\|_1, \quad v \in (H^1(\Omega))^d, \ w \in H^1(\Omega). \tag{5.1}$$

Set

$$e_h = \bar{P}_h(p - p_h).$$

Then, the error equation (4.1) can be written as

$$(K^{-1}(u - u_h), v) - (\mathrm{div}\, v, e_h) = 0 \quad \forall v \in V_h, \tag{5.2a}$$

$$(\mathrm{div}(u - u_h), w) - G_h(p_h, w) = 0 \quad \forall w \in W_h. \tag{5.2b}$$

Theorem 5.1 *Under assumptions* (2.10a), (2.10b), (2.10c)*, and* (5.1)*, it holds that*

$$\|e_h\|_0 \le Ch \left(\|u - u_h\|_0 + \|\mathrm{div}(u - u_h)\|_0\right). \tag{5.3}$$

Proof Define φ by

$$-\mathrm{div}(K\nabla\varphi) = e_h \quad \text{in } \Omega, \tag{5.4a}$$

$$\varphi = 0 \quad \text{on } \partial\Omega. \tag{5.4b}$$

The elliptic regularity implies (e.g., for a convex polygonal Ω)

$$\|\varphi\|_2 \le C\|e_h\|_0. \tag{5.5}$$

Then, by (3.2), (5.2a) and (5.2b), Green's formula, (5.4a) and (5.4b), and the fact that $G_h(p_h, \bar{P}_h\varphi) = 0$, we see that

$$\begin{aligned}
\|e_h\|_0^2 &= -(e_h, \mathrm{div}(K\nabla\varphi)) \\
&= -(e_h, \mathrm{div}\Pi_h[K\nabla\varphi]) \\
&= -(K^{-1}(u - u_h), \Pi_h[K\nabla\varphi]) \\
&= (K^{-1}(u - u_h), K\nabla\varphi - \Pi_h[K\nabla\varphi]) - (u - u_h, \nabla\varphi) \\
&= (K^{-1}(u - u_h), K\nabla\varphi - \Pi_h[K\nabla\varphi]) + (\mathrm{div}(u - u_h), \varphi) \\
&= (K^{-1}(u - u_h), K\nabla\varphi - \Pi_h[K\nabla\varphi]) + (\mathrm{div}(u - u_h), \\
&\qquad \varphi - \bar{P}_h\varphi),
\end{aligned}$$

which, together with (5.1) and (5.5), gives

$$\|e_h\|_0 \le Ch(\|u-u_h\| + \|\mathrm{div}(u-u_h)\|_0),$$

i.e., the desired estimate (5.3) holds. □

Example (continued) With the example given in (2.15), we obtain

$$\|e_h\|_0 \le Ch^{k+1}(\|u\|_k + \|\mathrm{div}\, u\|_k + \|p\|_k), \tag{5.6}$$

which shows a superconvergence result.

6 Local postprocessing

To improve the error estimate for p, a local postprocessing technique is used to obtain a new function $p_h^* \in W_h$, which is of more accurate approximation to the solution. As an example, we focus on the case where the triangulation T_h of Ω is a union of triangles or tetrahedra.

For each $T \in T_h$, following [13], define $p_h^* \in W_h$, element by element, as the solution of the equations

$$\int_T p_h^* dx = \int_T p_h dx, \tag{6.1a}$$

$$(K\nabla p_h^*, \nabla w)_T = (f,w)_T - \int_{\partial T} w u_h \cdot n_T \, d\ell \quad \forall w \in P_k(T), \tag{6.1b}$$

where n_T is the unit vector normal to ∂T and (u_h, p_h) is the solution of (2.14).

Theorem 6.1 *Under assumptions (2.10a), (2.10b), (2.10c), and (5.1), it holds that*

$$\|p-p_h^*\|_0 \le C\left(h^{k+1}|p|_{k+1} + h(\|u-u_h\|_0 + \|\mathrm{div}(u-u_h)\|_0)\right), \tag{6.2}$$

where p and p_h^* are the respective solutions of (2.2a) and (2.2b) and (6.1a) and (6.1b).

Proof Let P_T denote the L^2-projection onto $P_0(T)$. Note that $P_0(T) \subset W_h|_T$ for all finite element pairs considered so it follows from (2.13b) and the fact that $G_h(p_h, w) = 0$, $w \in P_0(T)$ that

$$(f,w)_T - \int_{\partial T} w u_h \cdot n_T \, d\ell = 0 \quad \forall w \in P_0(T),\ T \in T_h.$$

Therefore, systems (6.1a) and (6.1b) have a unique solution p_h^*.

Using (2.2a) and (2.5) yields

$$(K\nabla p, \nabla w)_T = (f,w)_T - \int_{\partial T} w u \cdot n_T \, d\ell \quad \forall w \in P_k(T),$$

so that the error equation for (6.1a) and (6.1b) is

$$(K\nabla(p-p_h^*), \nabla w)_T = \int_{\partial T} w(u_h - u)\cdot n_T \, d\ell \qquad \forall w \in P_k(T). \tag{6.3}$$

Take $\hat{p} \in P_k(T)$ to be the Neumann projection of the solution p on each $T \in T_h$, with the nonuniqueness constant on each element determined by $P_T(\hat{p} - p_h\|_T) = 0$.

Now, shifting p to p_h^* and choosing $w = \hat{p} - p_h^*$, we see that

$$(K\nabla(\hat{p}-p_h^*), \nabla(\hat{p}-p_h^*))_T = (K\nabla(\hat{p}-p), \nabla(\hat{p}-p_h^*))_T + \int_{\partial T} (\hat{p}-p_h^*)(u_h-u)\cdot n_T \, d\ell. \tag{6.4}$$

Hence, using (2.1), there is a constant $C > 0$ such that

$$\alpha_1 \|\nabla(\hat{p}-p_h^*)\|_{0,T}^2 \le C\|\nabla(\hat{p}-p)\|_{0,T}\|\nabla(\hat{p}-p_h^*)\|_{0,T} + \left(h_T \int_{\partial T} |(u_h-u)\cdot n_T|^2 d\ell\right)^{1/2} \times \left(\int_{\partial T} h_T^{-1}|\hat{p}-p_h^*|^2 d\ell\right)^{1/2}. \tag{6.5}$$

Due to $P_T(\hat{p} - p_h|_T) = 0$, a scaling argument yields

$$\left(h_T^{-1}\int_{\partial T} |\hat{p}-p_h^*|^2 d\ell\right)^{1/2} \le C\|\nabla(\hat{p}-p_h^*)\|_{0,T}, \tag{6.6}$$

and thus, (6.5) gives

$$\|\nabla(\hat{p}-p_h^*)\|_{0,T} \le C\left\{\|\nabla(\hat{p}-p)\|_{0,T} + \left(h_T\int_{\partial T} |(u_h-u)\cdot n_T|^2 d\ell\right)^{1/2}\right\}. \tag{6.7}$$

Again, it is easily proved by using a simple scaling argument that

$$\|w\|_{0,T} \le Ch_T\|\nabla w\|_{0,T},$$

for each $w \in (I - P_T)P_k(T)$. Consequently, it follows from (6.7) that

$$\|\hat{p}-p_h^*\|_{0,T} \le \|P_T(\hat{p}-p_h^*)\|_{0,T} + Ch_T\left\{\|\nabla(\hat{p}-p)\|_{0,T} + \left(\int_{\partial T} h_T|(u_h-u)\cdot n_T|^2 d\ell\right)^{1/2}\right\}. \tag{6.8}$$

We now want to estimate $\|P_T(\hat{p} - p_h^*)\|_{0,T}$. Since P_T is bounded, we have

$$\|P_T(\hat{p}-p_h^*)\|_{0,T} \le \|P_T(\hat{p}-p)\|_{0,T} + \|P_T(p-p_h^*)\|_{0,T} \le C\|\hat{p}-p\|_{0,T} + \|P_T(p-p_h^*)\|_{0,T}. \tag{6.9}$$

We note that, by the definition of P_T,

$$P_T\bar{P}_h|_T = P_T,$$

and thus, using (6.1a) and (6.9), we have

$$\|P_T(\hat{p}-p_h^*)\|_{0,T} \le C\left(\|\hat{p}-p\|_{0,T}+\|\bar{P}_h(p-p_h)\|_{0,T}\right). \tag{6.10}$$

Hence, it follows from (6.8)–(6.10) that

$$\|\hat{p}-p_h^*\|_{0,T} \le C\Bigg\{\|\hat{p}-p\|_{0,T}+\|\bar{P}_h(p-p_h)\|_{0,T} + h_T\left[\|\nabla(\hat{p}-p)\|_{0,T}+\left(h_T\int_{\partial T}|(u_h-u)\cdot n_T|^2 d\ell\right)^{1/2}\right]\Bigg\}. \tag{6.11}$$

For the last term in (6.11), note that

$$\left(h_T\int_{\partial T}|(u_h-\Pi_h u)\cdot n_T|^2 d\ell\right)^{1/2} \le C\|u_h-\Pi_h u\|_{0,T}.$$

Consequently, by the triangle inequality and the optimal approximation of $\Pi_h u$ to u, we obtain

$$\|\hat{p}-p_h^*\|_0 \le C(h^{k+1}|p|_{k+1}+\|\bar{P}_h(p-p_h)\|_0+h\|u_h-u\|_0). \tag{6.12}$$

Finally, the desired result (6.2) follows immediately from (6.12), Theorem 5.1, and the triangle inequality. The proof has been completed. □

Example (continued) With the example given in (2.15), we obtain

$$\|p-p_h^*\|_0 \le Ch^{k+1}(\|u\|_{k+1}+\|\text{div } u\|_k+\|p\|_k), \tag{6.13}$$

which obviously improves estimate (4.4) in the convergence rate for the pressure. Note that schemes (6.1a) and (6.1b) are locally defined and can thus be implemented in a parallel fashion.

7 Coupled Darcy–Stokes flow and its approximation

For the simplicity of presentation, let Ω be a polygonal domain in $\Re^2$, divided into two subdomains Ω_1 and Ω_2, with interface $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$. Define $\Gamma_i = \partial\Omega_i \setminus \Gamma_{12}$ for $i = 1, 2$. Denote by n the outward unit normal vector to $\partial\Omega$ and n_{12} (respectively, τ_{12}) the unit normal (respectively, tangential) vector to Γ_{12} outward to Ω_1 (see Fig. 1).

We assume that the flow satisfies the Stokes equations on Ω_1 and the single-phase Darcy equations on Ω_2 (both stationary):

$$-\nabla\cdot\hat{T}(u_1,p_1) = f_1 \qquad \text{in } \Omega_1, \tag{7.1}$$

$$\nabla\cdot u_1 = 0 \qquad \text{in } \Omega_1, \tag{7.2}$$

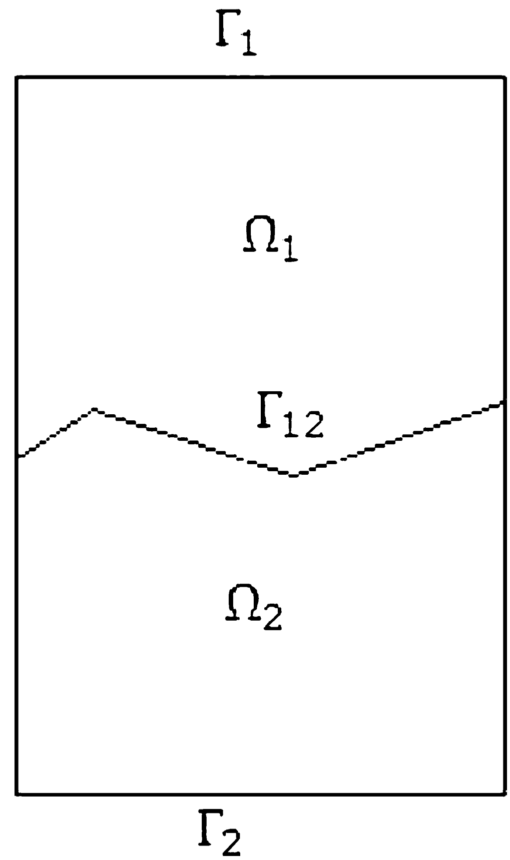

Fig. 1 A model problem

$$u_1 = 0 \qquad \text{on } \Gamma_1, \tag{7.3}$$

where $\hat{T}$ is the stress tensor

$$\hat{T}(u_1,p_1) = -p_1 I + 2\mu D(u_1)$$

which depends on the viscosity $\nu > 0$ and the strain tensor

$$D(u_1) = \frac{1}{2}(\nabla u_1 + \nabla u_1^T),$$

and

$$u_2 + K\nabla p_2 = 0 \qquad \text{in } \Omega_2, \tag{7.4}$$

$$\nabla\cdot u_2 = f_2 \qquad \text{in } \Omega_2, \tag{7.5}$$

$$u_2\cdot n = 0 \qquad \text{on } \Gamma_2, \tag{7.6}$$

$$\int_{\Omega_1} p_1\,dx + \int_{\Omega_2} p_2\,dx = 0, \tag{7.7}$$

$$u_1\cdot n_{12} = u_2\cdot n_{12} \qquad \text{on } \Gamma_{12}, \tag{7.8}$$

$$p_1 - 2\mu((D(u_1)n_{12})\cdot n_{12}) = p_2 \qquad \text{on } \Gamma_{12}, \tag{7.9}$$

$$u_1\cdot\tau_{12} = -2G(D(u_1)n_{12})\cdot\tau_{12} \qquad \text{on } \Gamma_{12}, \tag{7.10}$$

where K is a symmetric and positive definite tensor representing the fluid mobility that is assumed to satisfy (2.1) in Ω_2.

Note that condition (7.8) represents mass conservation across the interface, condition (7.9) imposes a balance of forces across the interface, and condition (7.10) is the Beavers–Joseph–Saffman law, where $G > 0$ is a friction constant that can be determined experimentally [2].

The Sobolev spaces $H^k(S) = W^{2,k}(S)$ are defined in the usual way for $S = \Omega_1$ or Ω_2 with the norm and seminorm $\|\cdot\|_{k,S}$ and $|\cdot|_{k,S}$, respectively (Ω_1 and Ω_2 are often omitted when there is no ambiguity). Let

$$X_1 = \{v_1 \in (H^1(\Omega_1))^2 : v_1 = 0 \text{ on } \Gamma_1\}, \quad W_1 = L^2(\Omega_1).$$

The norms in X_1 and W_1 are given by

$$\|v_1\|_{X_1}^2 = \|\nabla v_1\|_{0,\Omega_1}^2 + \frac{\mu}{G} \sum_{e\in\Gamma_{12}} \|v_1 \cdot \tau_{12}\|_{0,e}^2,$$
$$\|p_1\|_{W_1} = \|p_1\|_{0,\Omega_1}, \quad v_1 \in X_1,\ p_1 \in W_1.$$

The velocity–pressure spaces on Ω_2 are

$$X_2 = \left\{v \in H(\text{div};\Omega_2) : \int_{\partial\Omega_2} v \cdot nw\, d\ell = 0 \quad \forall\, w \in H^1_{0,\Gamma_{12}}(\Omega_2)\right\},$$
$$W_2 = L^2(\Omega_2),$$

where

$$H(\text{div};\Omega_2) = \{v_2 \in (L^2(\Omega_2))^2 : \nabla\cdot v_2 \in L^2(\Omega_2)\},$$
$$H^1_{0,\Gamma_{12}}(\Omega_2) = \{w \in H^1(\Omega_2) : w = 0 \quad \text{on } \Gamma_{12}\}.$$

The norms associated with (X_2, W_2) are

$$\|v_2\|_{X_2}^2 = \|v_2\|_{0,\Omega_2}^2 + \|\nabla\cdot v_2\|_{0,\Omega_2}^2,$$
$$\|p_2\|_{W_2} = \|p_2\|_{0,\Omega_2}, \quad v_2 \in X_2,\ p_2 \in W_2.$$

We can now define the space $X = X_1 \times X_2$ with the norm

$$\|v\|_X = \left(\|v_1\|_{X_1}^2 + \|v_2\|_{X_2}^2\right)^{1/2} \qquad \forall\, v \in X,$$

and the space

$$W = \big\{p = (p_1, p_2) : p_i \in W_i,\ i = 1,2, \text{ and } (p_1, 1)_{\Omega_1} + (p_2, 1)_{\Omega_2} = 0\big\},$$

with the norm

$$\|p\|_W = \left(\|p_1\|_{W_1}^2 + \|p_2\|_{W_2}^2\right)^{1/2}.$$

We define the bilinear forms $a_1(\cdot,\cdot)$ and $b_1(\cdot,\cdot)$ on $X_1 \times X_1$ and $X_1 \times W_1$, respectively, by

$$a_1(u_1, v_1) = 2\mu \int_{\Omega_1} D(u_1) : D(v_1)\, dx + \frac{\mu}{G} \sum_{e\in\Gamma_{12}} \int_e u_1 \cdot \tau_{12} v_1 \cdot \tau_{12}\, d\ell,$$

$$b_1(v_1, p_1) = -(\nabla\cdot v_1, p_1) = -\int_{\Omega_1} p_1 \nabla\cdot v_1\, dx,$$

and the bilinear forms $a_2(\cdot,\cdot)$ and $b_2(\cdot,\cdot)$ on $X_2 \times X_2$ and $X_2 \times W_2$, respectively, by

$$a_2(u_2, v_2) = \int_{\Omega_2} K^{-1} u_2 \cdot v_2\, dx,$$
$$b_2(v_2, p_2) = -(\nabla\cdot v_2, p_2) = -\int_{\Omega_2} p_2 \nabla\cdot v_2\, dx.$$

Defining $a = a_1 + a_2$ and $b = b_1 + b_2$, the variational formulation for the coupled Darcy–Stokes flow problem is to find $(u, p) \in X \times W$ such that

$$a(u, v) + b(v, p) - b(u, q) + \int_{\Gamma_{12}} p_2(v_1 - v_2)\cdot n_{12}\, d\ell = \int_{\Omega_1} f_1 v_1 dx + \int_{\Omega_2} f_2 q_2\, dx \quad \forall\, (v, q) \in X \times W. \tag{7.11}$$

Let h_i be a positive parameter and T_h^i be a regular triangulation of Ω_i into triangles $\{T_j^i\}$ such that $\overline{\Omega_i} = \cup \overline{T_j^i}$ [14, 16], $i = 1, 2$. To simplify the notation, we assume that the cells $T \in T_h^i$ are affine equivalent, the grids T_h^1 and T_h^2 match at Γ_{12}, Γ_{12} is polygonal, and no point of the interface boundary $\partial\Gamma_{12}$ belongs to the interior of an element edge. Then, we define their respective finite-dimensional subspaces of X and W as X_h and W_h:

$$X_h = X_h^1 \times X_h^2, \quad W_h = W_h^1 \times W_h^2,$$

where

$$X_h^1 = \left\{v_1 \in X_1 : v_1|_T \in P_1(T) \quad \forall\, T \in T_h^1\right\},$$
$$W_h^1 = \left\{p_1 \in H^1(\Omega_1) : p_1|_T \in P_1(T) \quad \forall\, T \in T_h^1\right\},$$
$$X_h^2 = \left\{v_2 \in X_2 : v_2|_T \in P_1(T) \quad \forall\, T \in T_h^2,\ v_2\cdot n = 0 \text{ on } \Gamma_2\right\},$$
$$W_h^2 = \left\{p_2 \in W_2 : p_2|_T \in P_1(T) \quad \forall\, T \in T_h^2\right\}.$$

We also need another two spaces:

$$\bar W_h^1 = \{p_1 \in W_1 : p_1|_T \in P_0(T) \quad \forall\, T \in T_h^1\},$$
$$\bar W_h^2 = \{p_2 \in W_2 : p_2|_T \in P_0(T) \quad \forall\, T \in T_h^2\}.$$

As in Section 2, we define the L^2-projection operator $\bar P_h : L^2(\Omega) \to \bar W_h$ by

$$(p, q_h) = (\bar P_h p, q_h) \quad \forall p \in L^2(\Omega),\ q_h \in \bar W_h, \tag{7.12}$$

where $\bar W_h = \bar W_h^1 \times \bar W_h^2 \subset L^2(\Omega)$ denotes the piecewise constant space associated with $T_h(\Omega)$.

The projection operator $\bar P_h$ has the following properties:

$$\|\bar P_h p\|_0 \le C\|p\|_0 \quad \forall p \in L^2(\Omega), \tag{7.13}$$

$$\|p - \bar P_h p\|_0 \le Ch\|p\|_1 \quad \forall p \in H^1(\Omega). \tag{7.14}$$

Now, using (7.12), as in (2.12), we can define the bilinear form $G_h(\cdot,\cdot)$ as follows:

$$G_h(p,q) = (p-\bar{P}_h p, q) = (p-\bar{P}_h p, q-\bar{P}_h q), \quad p,q \in L^2(\Omega). \tag{7.15}$$

Define the finite-dimensional space of functions on the interface $\wedge_h = X_h^2 \cdot n_{12}$ and let

$$V_h = \left\{ v = (v_1, v_2) \in X_h : \sum_{e\in\Gamma_{12}} \int_e \eta(v_1 - v_2)\cdot n_{12}\, d\ell = 0 \quad \forall \eta \in \wedge_h \right\}.$$

Then, the finite element approximation of the coupled problem is to find a pair $(u_h, p_h) \in (V_h, W_h)$ such that

$$a(u_h, v_h) + b(v_h, p_h) - b(u_h, q_h) + G_h(p_h, q_h) = (f, v_h) + (g, q_h) \quad \forall\, (v,q) \in V_h \times W_h,$$

where $(f, v_h) = (f_1, v_1^h)$ and $(g, q_h) = (f_2, q_2^h)$. That is,

$$\mathcal{B}((u_h, p_h), (v_h, q_h)) = (f, v_h) + (g, q_h) \quad \forall (v_h, q_h) \in (V_h, W_h), \tag{7.16}$$

where

$$\mathcal{B}((u_h, p_h), (v_h, q_h)) = a(u_h, v_h) + b(v_h, p_h) - b(u_h, q_h) + G_h(p_h, q_h).$$

8 Stability of the approximation method

The stability result of the stabilized finite element method (7.16) is shown in this section. We will make use of the quasi-local interpolant $\Pi_h^1 : (H^1(\Omega_1))^2 \to X_h^1$ satisfying for all $v_1 \in (H_1(\Omega_1))^2$,

$$b_1(\Pi_h^1 v_1 - v_1, q_1) = 0 \quad \forall\, q_1 \in \bar{W}_h^1,$$
$$\|\Pi_h^1 v_1\|_{X^1} \le C\|v_1\|_{X^1},$$
$$\|\Pi_h^1 v_1 - v_1\|_{X^1} \le Ch_1\|v_1\|_2,$$

and the linear operator $\Pi_h^2 : (H^1(\Omega_2))^2 \to X_h^2$ satisfying for all $v_2 \in (H_1(\Omega_2))^2$,

$$b_2(\Pi_h^2 v_2 - v_2, q_2) = 0 \quad \forall\, q_2 \in \bar{W}_h^2,$$
$$\|\Pi_h^2 v_2\|_{X^2} \le C\|v_2\|_{X^2},$$
$$\|\Pi_h^2 v_2 - v_2\|_{X^2} \le Ch_2\|v_2\|_2.$$

In addition, define $R_h = (R_h^1, R_h^2) : X_1 \times (X_2 \cap (H^1(\Omega_2))^2) \to V_h$, with $R_h v = (R_h^1 v_1, R_h^2 v_2)$, satisfying

$$b(R_h v - v, q_h) = 0 \quad \forall q_h \in \bar{W}_h,$$
$$\|R_h v\|_X \le C\|v\|_{1,\Omega},$$
$$\|v - R_h v\|_X \le C(h_1\|v_1\|_2 + h_2\|v_2\|_2), \tag{8.1}$$

where R_h^1 and R_h^2 are constructed as in [28]. Finally, define $P_h = (P_h^1, P_h^2) : W \to W_h$ satisfying

$$(p - P_h p, q) = 0, \quad p \in W, q \in W_h,$$

and

$$\|p_1 - P_h^1 p_1\|_0 \le Ch_1\|p_1\|_1,$$
$$\|p_2 - P_h^2 p_2\|_0 \le Ch_2\|p_2\|_1.$$

That is,

$$\|p - P_h p\|_0 \le C(h_1\|p_1\|_1 + h_2\|p_2\|_1), \quad p = (p_1, p_2) \in W. \tag{8.2}$$

Theorem 8.1 *Under condition (2.1), the bilinear form $\mathcal{B}((\cdot,\cdot),(\cdot,\cdot))$ satisfies the continuous property*

$$|\mathcal{B}((u_h, p_h), (v_h, q_h))| \le C(\|u_h\|_X + \|p_h\|_W)(\|v_h\|_X + \|q_h\|_W) \quad \forall\, (u_h, p_h), (v_h, q_h) \in (V_h, W_h) \tag{8.3}$$

and the coercive property

$$\sup_{0\neq(v_h,q_h)\in(V_h,W_h)} \frac{|\mathcal{B}((u_h, p_h), (v_h, q_h))|}{\|v_h\|_X + \|q_h\|_W} \ge \beta(\|u_h\|_X + \|p_h\|_W) \quad \forall\, (u_h, p_h) \in (V_h, W_h), \tag{8.4}$$

where the constant $\beta > 0$ is independent of $h_i (i = 1, 2)$.

Proof From the definition of $\mathcal{B}((\cdot,\cdot),(\cdot,\cdot))$, $\|\cdot\|_X$ and $\|\cdot\|_W$, the continuous property (8.3) can be easily proven. It is sufficient to show the coercive property (8.4).

For any $p_h \in W_h \subset L_0^2(\Omega)$, there exists $v \in (H^1(\Omega))^2$ such that

$$\nabla \cdot v = -p_h,$$

satisfying

$$\|v\|_{1,\Omega} \le C\|p_h\|_{0,\Omega}.$$

Let $(v_h, q_h) = (u_h - \alpha R_h v, p_h + \beta\nabla\cdot u_h)$ in (7.16); we have

$$\mathcal{B}((u_h, p_h), (u_h - \alpha R_h v, p_h + \beta\nabla\cdot u_h)) = a(u_h, u_h) - \alpha a(u_h, R_h v) - \alpha b(R_h v, p_h) - \beta b(u_h, \nabla\cdot u_h) + G_h(p_h, p_h) + \beta G_h(p_h, \nabla\cdot u_h).$$

Note that, by (2.1),

$$a(u_h, u_h) \ge C_0\|u_h\|_X^2,$$

for some constant $C_0 > 0$. Also, we see that

$$|\alpha a(u_h, R_h v)| \le \alpha\|u_h\|_X\|R_h v\|_X \le \frac{C_0}{2}\|u_h\|_X^2 + C_1'\alpha^2\|R_h v\|_X^2 \le \frac{C_0}{2}\|u_h\|_X^2 + C_1\alpha^2\|p\|_0^2,$$

$$|\beta G_h(p_h, \nabla\cdot u_h)| \le \beta G_h^{1/2}(p_h, p_h)\|\nabla\cdot u_h\|_0 \le \frac{1}{2}G_h(p_h, p_h) + C_2\beta^2\|\nabla\cdot u_h\|_0^2.$$

Also, note that

$$\begin{aligned}\|p_h\|_0^2 &= -(p_h, \nabla\cdot v)\\ &= -(p_h - \bar{P}_h p_h, \nabla\cdot v) - (\bar{P}_h p_h, \nabla\cdot v)\\ &= -(p_h - \bar{P}_h p_h, \nabla\cdot v) - (\bar{P}_h p_h, \nabla\cdot R_h v)\\ &= -(p_h - \bar{P}_h p_h, \nabla\cdot (v - R_h v)) - (p_h, \nabla\cdot R_h v)\\ &\le |(p_h - \bar{P}_h p_h, \nabla\cdot (v - R_h v))| + |(\bar{P}_h p_h, \nabla\cdot R_h v)|,\end{aligned}$$

and

$$\begin{aligned}|(p_h - \bar{P}_h p_h, \nabla\cdot (v - R_h v))| &\le C G_h^{\frac{1}{2}}(p_h, p_h)\|\nabla \cdot (v - R_h v)\|_0\\ &\le C G_h^{\frac{1}{2}}(p_h, p_h)\|p_h\|_0\\ &\le \frac{1}{2}\|p_h\|_0^2 + C_3 G_h(p_h, p_h).\end{aligned}$$

Then, we have

$$|\alpha b(R_h v, p_h)| \ge \frac{\alpha}{2}\|p_h\|_0^2 - C_3\alpha G_h(p_h, p_h).$$

Hence, combining all the above inequalities yields

$$\begin{aligned}&\mathcal{B}((u_h, p_h), (u_h - \alpha R_h v, p_h + \beta\nabla\cdot u_h))\\ &\ge \frac{C_0}{2}\|u_h\|_X^2 + \alpha\left(\frac{1}{2} - C_1\alpha\right)\|p_h\|_0^2 + \left(\frac{1}{2} - C_3\alpha\right)\\ &\quad\times G_h(p_h, p_h) + \beta(\beta - C_2)\|\nabla\cdot u_h\|_0^2.\end{aligned}$$

If we choose appropriate constants $\alpha > 0$ and $\beta > 0$ such that

$$\frac{1}{2} - C_1\alpha \ge C_4 > 0,\ \frac{1}{2} - C_3\alpha \ge C_5 > 0,\ \beta - C_2 \ge C_6 > 0,$$

then

$$\mathcal{B}((u_h, p_h), (u_h - \alpha R_h v, p_h + \beta\nabla\cdot u_h)) \ge C(\|u_h\|_X + \|p_h\|_W)^2. \tag{8.5}$$

It is clear that

$$\begin{aligned}\|v_h\|_X + \|q_h\|_W &= \|u_h - \alpha R_h v\|_X + \|p_h + \beta\nabla\cdot u_h\|_W\\ &\le C(\|u_h\|_X + \|p_h\|_W).\end{aligned} \tag{8.6}$$

Finally, combining (8.5) and (8.6) completes the proof of (8.4). □

From Theorem 8.1, the following theorem follows:

Theorem 8.2 *Under condition (2.1), there exists a unique solution pair (u_h, p_h) to equation (7.16).*

9 Error estimates of the approximation method

The error estimate for the approximation method (7.16) is given in terms of the energy norm for u and p.

Theorem 9.1 *Assume that $u|_{\Omega_i} \in H^2(\Omega_i)$ and $p|_{\Omega_i} \in H^1(\Omega_i)$ for $i = 1, 2$ and condition (2.1) holds. Let $(u, p) \in X \times W$ be the solution of the coupled problem (7.1)–(7.10) and $(u_h, p_h) \in V_h \times W_h$ be the discrete solution of (7.16); then, the following estimate holds*

$$\begin{aligned}\|u - u_h\|_X + \|p - p_h\|_W \le C\Big\{&(h_1(\|u_1\|_{2,\Omega_1} + \|p_1\|_{1,\Omega_1})\\ &+h_2(\|u\|_{2,\Omega_2} + \|p_2\|_{1,\Omega_2}) + h_1^{1/2}h_2^{1/2}\|p_2\|_{1,\Omega_2}\Big\}.\end{aligned}$$

Proof From (7.11) and (7.16), the error equation is

$$\begin{aligned}&a(u - R_h u, v_h) + b(v_h, p - P_h p) - b(u - R_h u, q_h)\\ &+ \int_{\Gamma_{12}} p_2(v_1^h - v_2^h)\cdot n_{12}\, d\ell - G_h(p_h, q_h)\\ &= a(u_h - R_h u, v_h) + b(v_h, p_h - P_h p) - b(u_h - R_h u, q_h).\end{aligned}$$

Define $e = u_h - R_h u$ and $\eta = p_h - P_h p$. From Theorem 8.1 and the above equation, we have

$$\begin{aligned}\beta(\|e\|_X + \|\eta\|_W) &\le \sup_{0\ne(v_h,q_h)\in(V_h,W_h)} \frac{|\mathcal{B}((e,\eta),(v_h,q_h))|}{\|v_h\|_X + \|q_h\|_W}\\ &= \sup_{0\ne(v_h,q_h)\in(V_h,W_h)} |a(u - R_h u, v_h)\\ &\quad + b(v_h, p - P_h p) - b(u - R_h u, q_h)\\ &\quad + \int_{\Gamma_{12}} p_2\left(v_1^h - v_2^h\right)\cdot n_{12}\, d\ell\\ &\quad - G_h(P_h p, q_h)|/(\|v_h\|_X + \|q_h\|_W).\end{aligned}$$

From the standard interpolation theory, (8.1), (8.2), and the continuity of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, we see that

$$\begin{aligned}a(u - R_h u, v_h) &= a_1(u - R_h u, v_h) + a_2(u - R_h u, v_h)\\ &\le C(h_1\|u_1\|_2 + h_2\|u_2\|_2)(\|v_h\|_X + \|q_h\|_W),\\ b(v_h, p - P_h p) &\le C(h_1\|p_1\|_1 + h_2\|p_2\|_1)(\|v_h\|_X + \|q_h\|_W),\\ b(u - R_h u, q_h) &\le C(h_1\|u_1\|_2 + h_2\|u_2\|_2)(\|v_h\|_X + \|q_h\|_W),\\ G_h(P_h p, q_h) &\le C(h_1\|p_1\|_1 + h_2\|p_2\|_1)(\|v_h\|_X + \|q_h\|_W).\end{aligned}$$

Since $v_h \in V_h$, we have

$$\begin{aligned}&\int_{\Gamma_{12}} p_2\left(v_1^h - v_2^h\right)\cdot n_{12}\, d\ell\\ &= \sum_{e\in\Gamma_{12}}\int_e \left(p_2 - p_2^e\right)\left(v_1^h - v_2^h\right)\cdot n_{12}\, d\ell,\end{aligned}$$

where $p_2^e \in \wedge_h$ is the L^2-projection of p_2 with respect to the L^2-inner product on the edge e. By the definition of the projection, since $\wedge_h = X_h^2\cdot n_{12}$, we have

$$\sum_{e\in\Gamma_{12}}\int_e \left(p_2 - p_2^e\right) v_2^h\cdot n_{12}\, d\ell = 0.$$

Also, note that for any edge e and any constant vector c_e, we have

$$\sum_{e\in\Gamma_{12}} \int_e (p_2 - p_2^e)\, v_1^h \cdot n_{12}\, d\ell = \sum_{e\in\Gamma_{12}} \int_e (p_2 - p_2^e)\left(v_1^h - c_e\right) \cdot n_{12}\, d\ell \le \sum_{e\in\Gamma_{12}} \| (p_2 - p_2^e\|_{0,e} \| v_1^h - c_e\|_{0,e}.$$

Assume that each edge e of Γ_{12} is shared by the element $E_e^2 \in T_h^2$ and parts of the elements $E_{e,i}^1 \in T_h^1$, $i = 1, 2, \ldots, n_e$. Then, from the approximation properties and the trace inequality, we obtain

$$\int_e (p_2 - p_2^e)\, v_1^h \cdot n_{12}\, d\ell \le C h_2^{3/2} \|p_2\|_{2,E_e^2} \sum_{i=1}^{n_e} \left(h_1^{-1/2} \|v_1^h - c_e\|_{0,E_{e,i}^1} + h_1^{1/2} \|\nabla v_1^h\|_{0,E_{e,i}^1} \right).$$

Therefore, we see that

$$\sum_{e\in\Gamma_{12}} \int_e (p_2 - p_2^e) v_1^h \cdot n_{12}\, dl \le C \sum_{e\in\Gamma_{12}} C h_2^{3/2} \|p_2\|_{2,E_e^2} \sum_{i=1}^{n_e} h_1^{1/2} \|\nabla v_1^h\|_{0,E_{e,i}^1} \le C h_1^{1/2} h_2^{1/2} \|p_2\|_{2,\Omega_2} \|v_1^h\|_{1,\Omega_1}.$$

Combining all the above equations, we obtain the desired result. □

10 Numerical results

Numerical results are presented to check the theory developed in the previous sections for the Darcy flow; preliminary numerical results for the coupled Darcy–Stokes flow were reported in [31]. In all the experiments, the triangulations T_h are based on the partition of the unit square $\Omega = [0, 1] \times [0, 1]$ into triangles. The exact solution for the velocity $u = (u_1, u_2)$ and the pressure p is given as follows:

$$p(x) = \sin(2\pi x_1)\sin(2\pi x_2), \quad x = (x_1, x_2),$$
$$u_1(x) = -2\pi \cos(2\pi x_1)\sin(2\pi x_2),$$
$$u_2(x) = -2\pi \sin(2\pi x_1)\cos(2\pi x_2).$$

The coefficient tensor K is the identity tensor, and the right-hand side f is determined by (2.2a) through this exact solution.

Example 1 In the first example, we test the case $k = 1$ for the error estimates $\|p - p_h\|_0$, $\|u - u_h\|_0$, $\|\text{div}(u - u_h)\|_0$, $\|\bar{P}_h(p - p_h)\|_0$, and $\|p - p_h^*\|_0$, which are reported in Table 1 (with the respective notation p_{L^2}, u_{L^2}, u_{div}, $\bar{P}_{hL^2}$, and p_h^*). It can be seen that all these computational estimates agree with the theoretical estimates (4.4), (5.6), and (6.13) except for $\|p - p_h\|_0$ and $\|u - u_h\|_0$; the computational estimates are better.

As a comparison with a comparable element that satisfies the inf–sup condition, we consider the same problem for the lowest-order Brezzi–Douglas–Marini element [7]. The computational results are displayed in Table 2. It follows from this table that the error estimate for the pressure using this element is less accurate than that presented in Table 1. The computational times for these two elements are the same.

Example 2 In the second example, we test the case $k = 2$ for the same error estimates as those reported in Example 1,

Table 2 Error estimates for the lowest-order Brezzi–Douglas–Marini element

$1/h$	u_{L^2} rate	u_{div} rate	p_{L^2} rate
8			
16	1.94471	0.979296	1.01198
24	1.98171	0.993411	1.00614
32	1.99072	0.996746	1.00332
40	1.99433	0.998057	1.00205
48	1.99615	0.998708	1.00138

Table 1 Error estimates for the pair $P_1 - P_1$

$1/h$	u_{L^2} rate	u_{div} rate	p_{L^2} rate	$\bar{P}_{hL^2}$ rate	p_h^* rate
8					
16	2.02470	0.973923	2.25806	2.47020	2.14625
24	1.94496	0.992501	2.07460	2.18598	2.05022
32	1.93170	0.996590	2.03650	2.10075	2.02583
40	1.92833	0.998092	2.02182	2.06370	2.01584
48	1.92760	0.998798	2.01460	2.04416	2.01075

Table 3 Error estimates for the pair $P_2 - P_2$

$1/h$	u_{L^2} rate	u_{div} rate	p_{L^2} rate	$\bar{P}_{hL^2}$ rate	p_h^* rate
8					
16	2.06083	1.97673	2.06355	3.78094	3.04706
24	2.02310	1.99206	2.02217	3.75161	3.01579
32	2.01218	1.99592	2.01139	3.71595	3.00821
40	2.00749	1.99749	2.00692	3.68760	3.00503
48	2.00506	1.99829	2.00465	3.66541	3.00340

which are illustrated in Table 3. This time, all the computational estimates now agree with the theoretical estimates (4.4), (5.6), and (6.13). In particular, this example shows that the estimate (4.4) is generally the best estimate one can obtain with the stabilized mixed finite element methods (2.13a) and (2.13b). Numerical results for the $P_2 - P_2$ pair were also reported in the computational studies [5, 11]; however, only the first-order convergence rate was reported for the estimate $\|\mathrm{div}(u - u_h)\|_0$. Continuous finite elements were used in [5]. If the estimate $\|u - u_h\|_0$ is first obtained and $\|\mathrm{div}(u - u_h)\|_0$ is then measured by applying the divergence operator, the accuracy of the latter estimate will be reduced. For the current stabilized mixed finite element method, the estimate $\|\mathrm{div}(u - u_h)\|_0$ must directly be determined since both estimates have the same accuracy.

11 Concluding remarks

In this paper, we have systematically studied the pressure projection stabilization method for the Darcy and coupled Darcy–Stokes flow problems in multiple dimensions. For the Darcy flow, while the analysis in this paper has been focused in (2.12), it can be extended to other stabilization terms using the equal-order mixed finite element pairs, as pointed out in [11]. Note that, by the definition of the L^2-projection $\bar{P}_h$,

$$\int_T (p_h - \bar{P}_h p_h)\, d\ell = 0, \quad T \in T_h.$$

Then, a simple scaling argument shows

$$\|p_h - \bar{P}_h p_h\|_{0,T} \le C\|h_T \nabla(p_h - \bar{P}_h p_h)\|_{0,T}, \quad T \in T_h.$$

Consequently, the stabilization term defined in (2.12) can be replaced by

$$G_h(p,q) = \sum_{T \in T_h} \left(h_T \nabla(p_h - \bar{P}_h p_h), h_T \nabla(p_h - \bar{P}_h p_h)\right)_T, \tag{11.1}$$

and all the analysis performed for (2.12) can be extended to the stabilization term (11.1). The latter term is equivalent to the stabilization term considered in [3, 10], where an interpolation operator was used in place of $\bar{P}_h$. The analysis in this paper can be extended to any stabilization term that is equivalent to either (2.12) or (11.1).

We mention that while the analysis has been performed only for the Dirichlet boundary condition (2.2b), it is also valid for the boundary conditions of the Neumann and mixed types. The Neumann boundary condition is an essential boundary condition that needs to be imposed in the vector space V for the mixed method, and the mixed condition is natural as the Dirichlet one [14, 16].

Finally, as an example, the $P_1 - P_1$ pair on triangles has been analyzed for the pressure projection stabilization method for the coupled Darcy–Stokes flow. The analysis can be also extended to other types of finite element pairs, such as higher-order elements.

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