Available online at www.sciencedirect.com

ScienceDirect

Comput. Methods Appl. Mech. Engrg. 197 (2007) 22–35

Computer methods in applied mechanics and engineering

www.elsevier.com/locate/cma

A new stabilized finite element method for the transient Navier–Stokes equations ☆

Jian Li [a,b], Yinnian He [a], Zhangxin Chen [a,c,*]

[a] Faculty of Science, Xi'an Jiaotong University, Xi'an 710049, PR China
[b] Department of Mathematics, Baoji University of Arts and Sciences, Baoji 721007, PR China
[c] Department of Chemical and Petroleum Engineering, Schulich School of Engineering, University of Calgary, 2500 University Drive N.W. Calgary, Alberta, Canada T2N 1N4

Received 25 September 2006; received in revised form 21 June 2007; accepted 29 June 2007
Available online 11 August 2007

Abstract

This paper is concerned with the development and analysis of a new stabilized finite element method based on two local Gauss integrations for the two-dimensional transient Navier–Stokes equations by using the lowest equal-order pair of finite elements. This new stabilized finite element method has some prominent features: parameter-free, avoiding higher-order derivatives or edge-based data structures, and stabilization being completely local at the element level. An optimal error estimate for approximate velocity and pressure is obtained by applying the technique of the Galerkin finite element method under certain regularity assumptions on the solution. Compared with other stabilized methods (using the same pair of mixed finite elements) for the two-dimensional transient Navier–Stokes equations through a series of numerical experiments, it is shown that this new stabilized method has better stability and accuracy results.

AMS(MOS) Subject Classification: 35Q10; 65N30; 76D05

Keywords: Navier–Stokes equations; Stabilized finite element method; *inf-sup* condition; Local Gauss integration; Error estimate; Numerical experiments; Stability

1. Introduction

The development of stable mixed finite element methods is a fundamental component in the search for efficient numerical methods for solving the Navier–Stokes equations governing the flow of an incompressible fluid in terms of their primitive variable formulation. The importance of ensuring the compatibility of the component approximations for velocity and pressure by satisfying an *inf-sup* condition is widely understood. Numerous mixed finite elements satisfying this condition have been proposed over the past years. However, mixed elements that do not satisfy this condition (termed unstable) may also work well. Some of the unstable elements are very attractive and useful on many occasions. In particular, the equal-order pairs of mixed finite elements for the velocity and pressure are of practical importance in scientific computation because they are computationally convenient and efficient in a parallel or multigrid context [26]. Hence much attention has been recently paid to the study of the equal-order pairs of mixed finite elements.

In order to use the equal-order pairs of mixed finite elements, a popular strategy is to introduce stabilization or penalty techniques to enforce the *inf-sup* compatibility condition [4,6,8,11,13,19,21,24]. A common drawback in these stabilization techniques is, however, that stabilization

☆ This work is supported in part by the NSF of China 10671154, 10701001, and by the US National Science Foundation Grant DMS-0609995.

* Corresponding author. Address: Department of Chemical and Petroleum Engineering, Schulich School of Engineering, University of Calgary, 2500 University Drive N.W. Calgary, Alberta, Canada T2N 1N4.

E-mail addresses: jiaaanli@gmail.com (J. Li), heyn@mail.xjtu.edu.cn (Y. He), zhachen@ucalgary.ca (Z. Chen).

0045-7825/$ - see front matter
doi:10.1016/j.cma.2007.06.029

parameters are necessarily incurred either explicitly or implicitly. In addition, some of these techniques are conditionally stable and are of suboptimal accuracy, depending upon the choice of the stabilization parameters with respect to the solution regularity [4,21]. Thus the development of mixed finite elements free of stabilization parameters has become increasingly important.

Stabilized mixed finite element methods are often developed using residuals of the momentum equation [3,11]. These residual terms must be formulated using mesh-dependent parameters, whose optimal values are usually unknown. Particularly, for the lowest equal-order pairs of mixed elements such as $P_1 - P_1$ and $Q_1 - Q_1$, pressure and velocity derivatives in the residual either vanish or are poorly approximated, causing difficulties in the application of consistent stabilization. Other stabilized mixed methods involve non-residual stabilization. Examples include local and global pressure jump formulations where the continuity equation is relaxed using the jumps of pressure across element interfaces [23,25]. This stabilization strategy requires edge-based data structures and a subdivision of grids into patches.

Regularization of a discrete Stokes formulation has been recently developed to overcome the problem of incompatible mixed approximations [20]. The central idea of this regularization in this article is to use two local Gauss integrations (the difference between the consistent and under-integrated Gauss integration) in the discrete formulation. Unlike penalty methods [5,12,14,22] to decouple pressure and velocity, this regularization aims to relax the continuity equation to enforce the *inf-sup* condition in incompatible mixed spaces. It does not require an approximation of derivatives and a specification of mesh-dependent parameters, and it always leads to symmetric problems. In addition, it is completely local at the element level, and no edge-based data structure is required. Consequently, the new stabilized method under consideration can be integrated in existing codes with very little additional coding effort.

This paper aims to extend the stabilized finite element method for the Stokes equations [20] to the two-dimensional transient Navier–Stokes equations. Using two local Gauss integral approximation, we first define this stabilized method for the lowest equal-order pair of mixed finite elements such as $P_1 - P_1$ or $Q_1 - Q_1$. Then we show its well-posedness and derive optimal error estimates. The results indicate that this method has a convergence rate of the same order as the usual Galerkin finite element method using the same pair of finite elements. Finally, we numerically compare this new method with other numerical methods such as the standard Galerkin method, penalty methods, the regular (Galerkin least squares-GLS) method, the multiscale enrich method, and the stable finite element $P_2 - P_1$ or mini-element $P_1 b - P_1$ methods. Numerical experiments show that the new method does not suffer the difficulties that arise when unstructured mesh must be used in other stabilized finite element methods for the two-dimensional transient Navier–Stokes problems, and it is superior to the other stabilized methods compared in terms of stability and convergence.

The remainder of this paper is organized as follows. In the next section, an abstract functional setting for the two-dimensional Navier–Stokes equations is given, together with some basic notation. The stabilized finite element method is stated in Section 3. Error estimates for the stabilized finite element solution are derived in Sections 4 and 5. In Section 6, a series of numerical experiments are given to illustrate the theoretical results. We conclude with a few remarks in the final section.

2. Function settings

Let Ω be a bounded domain in $\Re^2$, with a Lipschitz-continuous boundary Γ, satisfying a further condition stated in (A1) below. The transient Navier–Stokes equations are

$$u_t - \nu\Delta u + \nabla p + (u\cdot\nabla)u + \frac{1}{2}(\operatorname{div} u)u = f, \operatorname{div} u = 0, \quad (x,t) \in \Omega\times(0,T], \tag{2.1}$$

$$u(x,0) = u_0(x), \quad x\in\Omega, \quad u(x,t)|_\Gamma = 0, \quad t\in[0,T], \tag{2.2}$$

where $u = u(x, t) = (u_1(x,t), u_2(x,t))$ represents the velocity vector, $p = p(x,t)$ the pressure, $f = f(x,t)$ the prescribed body force, $\nu > 0$ the viscosity, $T > 0$ the final time, and $u_t = \frac{\partial u}{\partial t}$. The term $(\operatorname{div} u)u/2$ is introduced to ensure the dissipativity of Eq. (2.1) [27].

To introduce a variational formulation, set

$$X = \left(H_0^1(\Omega)\right)^2, \quad Y = (L^2(\Omega))^2,$$
$$M = L_0^2(\Omega) = \left\{q\in L^2(\Omega); \int_\Omega q\,dx = 0\right\},$$
$$V = \{v\in X : \operatorname{div} v = 0\}, \quad D(A) = (H^2(\Omega))^2 \cap V.$$

As noted, a further assumption on Ω is needed:

(A1) Assume that Ω is regular in the sense that the unique solution $(v,q)\in(X,M)$ of the steady Stokes problem

$$-\Delta v + \nabla q = g, \quad \operatorname{div} v = 0 \text{ in}\Omega, \quad v|_{\partial\Omega} = 0$$

for a prescribed $g\in Y$ exists and satisfies

$$\|v\|_2 + \|q\|_1 \leqslant c\|g\|_0,$$

where $c > 0$ is a constant depending only on Ω and $\|\cdot\|_i$ denotes the usual norm of the Sobolev space $H^i(\Omega)$ or $(H^i(\Omega))^2$ for $i = 0,1,2$. Below the constant $c > 0$ will depend at most on the data (ν, T, u_0, Ω).

We denote by $(\cdot,\cdot)$ and $\|\cdot\|_0$ the inner product and norm on $L^2(\Omega)$ or $(L^2(\Omega))^2$, as appropriate. The spaces $H_0^1(\Omega)$ and X are equipped with their usual scalar product and norm

$$((u,v)) = (\nabla u, \nabla v), \quad \|u\|_1 = ((u,u))^{1/2}.$$

(Due to the norm equivalence between $\|u\|_1$ and $\|\nabla u\|_0$ on $H_0^1(\Omega)$, we are using the same notation for them.) It is well known that for each $v\in X$ there hold the following inequalities:

$$\|v\|_{L^4} \leqslant 2^{1/4}\|v\|_0^{1/2}\|v\|_1^{1/2}, \quad \|v\|_0 \leqslant \gamma\|v\|_1, \tag{2.3}$$

where γ is a positive constant depending only on Ω and $\|v\|_{L^4} = \left(\int_\Omega v^4 \mathrm{d}x\right)^{1/4}$.

(**A2**) The initial velocity $u_0 \in D(A)$ and the body force $f(x,t) \in L^2(0,T;Y)$ are assumed to satisfy

$$\|u_0\|_2 + \left(\int_0^T \left(\|f\|_0^2 + \|f_t\|_0^2\right)\mathrm{d}t\right)^{1/2} \leqslant c.$$

The continuous bilinear forms $a(\cdot,\cdot)$ on $X \times X$ and $d(\cdot,\cdot)$ on $X \times M$ are, respectively, defined by

$$a(u,v) = \nu((u,v)) \quad \forall u,v \in X,$$
$$d(v,q) = -(v,\nabla q) = (q, \operatorname{div} v) \quad \forall v \in X, q \in M,$$

and the generalized bilinear form on $(X,M)\times(X,M)$ is given by

$$B((u,p);(v,q)) = a(u,v) - d(v,p) + d(u,q).$$

Then there hold the following estimates for the bilinear term $B((\cdot,\cdot);(\cdot,\cdot))$ [5,14]:

$$|B((u,p);(u,p)) = \nu\|u\|_1^2, \tag{2.4}$$
$$|B((u,p);(v,q))| \leqslant c(\|u\|_1 + \|p\|_0)(\|v\|_1 + \|q\|_0), \tag{2.5}$$
$$\beta_0(\|u\|_1 + \|p\|_0) \leqslant \sup_{(v,q)\in(X,M)} \frac{|B((u,p);(v,q))|}{\|v\|_1 + \|q\|_0}, \tag{2.6}$$

for all (u,p), $(v,q) \in (X,M)$, where the constant $\beta_0 > 0$ is independent of h.

Also, the trilinear term $b(\cdot,\cdot,\cdot)$ on $X \times X \times X$ is defined by

$$b(u,v,w) = ((u\cdot\nabla)v,w) + \frac{1}{2}((\operatorname{div} u)v,w)$$
$$= \frac{1}{2}((u\cdot\nabla)v,w) - \frac{1}{2}((u\cdot\nabla)w,v) \quad \forall u,v,w \in X.$$

It satisfies

$$b(u,v,w) = -b(u,w,v), |b(u,v,w)| + |b(w,v,u)| + |b(u,w,v)| \tag{2.7}$$
$$\leqslant c\|u\|_0^{1/2}\|u\|_1^{1/2}\left(\|v\|_1\|w\|_0^{1/2}\|w\|_1^{1/2} + \|v\|_0^{1/2}\|v\|_1^{1/2}\|w\|_1\right), \tag{2.8}$$

for all u, v, $w \in X$, and

$$|b(u,v,w)| + |b(v,u,w)| + |b(w,u,v)| \leqslant c\|u\|_1\|v\|_2\|w\|_0, \tag{2.9}$$

for all $u \in X, v \in D(A), w \in Y$.

The mixed variational form of (2.1) and (2.2) is to seek $(u,p) \in (X,M)$, $t > 0$, such that, for all $(v,q) \in (X,M)$,

$$(u_t,v) + B((u,p);(v,q)) + b(u,u,v) = (f,v), \tag{2.10}$$
$$u(0) = u_0. \tag{2.11}$$

For convenience, we recall the Gronwall Lemma that will be frequently used.

Lemma 2.1. ([22]). *Let $g(t)$, $\ell(t)$, and $\xi(t)$ be three nonnegative functions satisfying, for $t \in [0,T]$,*

$$\xi(t) + G(t) \leqslant c + \int_0^t \ell \,\mathrm{d}s + \int_0^t g\xi \,\mathrm{d}s,$$

where $G(t)$ is a nonnegative function on $[0,T]$. Then

$$\xi(t) + G(t) \leqslant \left(c + \int_0^t \ell \mathrm{d}s\right)\exp\left(\int_0^t g\mathrm{d}s\right). \tag{2.12}$$

The following result concerning the existence, uniqueness, and regularity of a global strong solution to the Navier–Stokes equations is presented under the assumptions (A1) and (A2).

Lemma 2.2. ([17]). *Assume that (A1) and (A2) hold. Then, for any given $T > 0$ there exists a unique solution (u,p) satisfying the following regularities*:

$$\sup_{0<t\leqslant T}\left(\|u(t)\|_2^2 + \|p(t)\|_1^2 + \|u_t(t)\|_0^2\right) \leqslant c, \tag{2.13}$$

$$\sup_{0<t\leqslant T}\tau(t)\|u_t\|_1^2 + \int_0^T \tau(t)\left(\|u_t\|_2^2 + \|p_t\|_1^2 + \|u_{tt}\|_0^2\right)\mathrm{d}t \leqslant c, \tag{2.14}$$

where $\tau(t) = \min\{1,t\}$.

3. Stabilized finite element method

For $h > 0$, we introduce finite-dimensional subspaces $(X_h,M_h) \subset (X,M)$, which are associated with K_h, a triangulation of Ω into triangles or quadrilaterals, assumed to be regular in the usual sense [9,10]. We assume that for the finite element spaces (X_h,M_h), the following approximation properties hold: For $(v,q) \in (D(A), H^1(\Omega) \cap M)$, there exist approximations $I_h v \in X_h$ and $\rho_h q \in M_h$ such that

$$\|v - I_h v\|_0 + h\|v - I_h v\|_1 \leqslant ch^2\|v\|_2, \tag{3.1}$$
$$\|q - \rho_h q\|_0 + h\|q - \rho_h q\|_1 \leqslant ch\|q\|_1, \tag{3.2}$$

where the L^2-projection $\rho_h: M \to M_h$ satisfies

$$(p - \rho_h p, q_h) = 0 \quad \forall p \in M, \quad q_h \in M_h.$$

We also assume that the inverse inequality holds [9,10]

$$\|\nabla v_h\|_0 \leqslant ch^{-1}\|v_h\|_0 \quad \forall v_h \in X_h. \tag{3.3}$$

This paper focuses on the analysis for the unstable velocity-pressure pair of the lowest equal-order finite elements:

$$X_h = \{v_h \in C^0(\Omega)^2 \cap X : v_h|_K \in R_1(K)^2 \forall K \in K_h\}$$

and

$$M_h = \{q_h \in C^0(\Omega) \cap M : q_h|_K \in R_1(K) \forall K \in K_h\},$$

where $R_1(K) = Q_1(K)$ if K is quadrilateral and $R_1(K) = P_1(K)$ if K is triangular.

It is well known that this lowest equal-order finite element pair does not satisfy the *inf-sup* condition. We define the following local difference between a consistent and an under-integrated mass matrices the stabilized formulation [20]

$$G(p_h, q_h) = p_i^T(M_k - M_1)q_j = p_i^T M_k q_j - p_i^T M_1 q_j.$$

Here, we set

$$p_i^T = [p_0, p_1, \dots, p_{N-1}]^T, \quad q_j = [q_0, q_1, \dots, q_{N-1}],$$

$$M_{ij} = (\phi_i, \phi_j), \quad p_h = \sum_{i=0}^{N-1} p_i \phi_i,$$

$$p_i = p_h(x_i), \quad \forall p_h \in M_h, \quad i, j = 0, 1, \dots N-1,$$

where ϕ_i is the basis function of the pressure on the domain Ω such that its value is one at node x_i and zero at other nodes; the symmetric and positive $M_k, k \geqslant 2$ and M_1 are pressure mass matrix computed by using k-order and 1-order Gauss integrations in each direction, respectively; Also, p_i and q_i, $i = 0, 1 \dots, N-1$ are the value of p_h and q_h at the node x_i. p_i^T is the transpose of the matrix p_i.

Let $\Pi_h: M \to R_0$ be the standard L^2-projection with the following properties [20]:

$$(p, q_h) = (\Pi_h p, q_h) \quad \forall p \in M, \quad q_h \in R_0, \tag{3.4}$$

$$\|\Pi_h p\|_0 \leqslant c\|p\|_0 \quad \forall p \in M, \tag{3.5}$$

$$\|p - \Pi_h p\|_0 \leqslant ch\|p\|_1 \quad \forall p \in H^1(\Omega) \cap M, \tag{3.6}$$

where $R_0 = \{q_h \in M : q_h|_K \text{ is a constant}, \forall K \in K_h\}$. Then we can rewrite the bilinear form $G(\cdot,\cdot)$ by

$$G(p, q) = (p - \Pi_h p, q - \Pi_h q). \tag{3.7}$$

Remark. The bilinear form $G(\cdot,\cdot)$ in (3.7) is a symmetric, semi-positive definite form generated on each local set K. This term can alleviate and offset the *inf-sup* condition. Some details will be explained in Theorem 3.1.

Using the above notation, the variational stabilized formulation of problem (2.10) and (2.11) reads: Find $(u_h, p_h) \in (X_h, M_h)$, $t \in [0, T]$, such that, for all $(v_h, q_h) \in (X_h, M_h)$,

$$(u_{ht}, v_h) + \mathscr{B}((u_h, p_h); (v_h, q_h)) + b(u_h, u_h, v_h) = (f, v_h), \tag{3.8}$$

$$u_h(0) = u_{0h}, \tag{3.9}$$

where u_{0h} is an approximation of u_0, and

$$\mathscr{B}((u_h, p_h); (v_h, q_h)) = a(u_h, v_h) - d(v_h, p_h) + d(u_h, q_h) + G(p_h, q_h),$$

is the new stabilized bilinear form. The following theorem establishes the weak coercivity of (3.8) for the equal-order finite element pair $R_1 - R_1$ (see [20]).

Theorem 3.1. *Let (X_h, M_h) be defined as above. Then there exists a positive constant β, independent of h, such that*

$$|\mathscr{B}((u, p); (v, q))| \leqslant c(\|u\|_1 + \|p\|_0)(\|v\|_1 + \|q\|_0)$$
$$\forall (u, p), (v, q) \in (X, M), \tag{3.10}$$

$$\beta(\|u_h\|_1 + \|p_h\|_0) \leqslant \sup_{(v_h, q_h) \in (X_h, M_h)} \frac{|\mathscr{B}((u_h, p_h); (v_h, q_h))|}{\|v_h\|_1 + \|q_h\|_0}$$
$$\forall (u_h, p_h) \in (X_h, M_h), \tag{3.11}$$

$$|G(p, q)| \leqslant c\|p - \Pi_h p\|_0 \|q - \Pi_h q\|_0 \quad \forall p, q \in M. \tag{3.12}$$

4. Error analysis

To derive error estimates for the finite element solution (u_h, p_h), we also define the projection operator (R_h, Q_h): $(X, M) \to (X_h, M_h)$ by

$$\mathscr{B}((R_h(v, q), Q_h(v, q)); (v_h, q_h)) = B((v, q); (v_h, q_h))$$
$$\forall (v, q) \in (X, M), (v_h, q_h) \in (X_h, M_h), \tag{4.1}$$

which are well defined and satisfy the following approximation properties:

Lemma 4.1. *Under the assumptions of Theorem 3.1, the projection operator (R_h, Q_h) satisfies*

$$\|v - R_h(v, q)\|_1 + \|q - Q_h(v, q)\|_0 \leqslant c(\|v\|_1 + \|q\|_0), \tag{4.2}$$

for all $(v, q) \in (X, M)$ and

$$\|v - R_h(v, q)\|_0 + h(\|v - R_h(v, q)\|_1 + \|q - Q_h(v, q)\|_0)$$
$$\leqslant ch^2(\|v\|_2 + \|q\|_1), \tag{4.3}$$

for all $(v, q) \in (D(A), H^1(\Omega) \cap M)$.

Proof. First, using the triangle inequality, (2.5), (3.10)–(3.12) and (4.1) gives

$$\|v - R_h(v, q)\|_1 + \|q - Q_h(v, q)\|_0$$
$$\leqslant \|v\|_1 + \|q\|_0 + \|R_h(v, q)\|_1 + \|Q_h(v, q)\|_0$$
$$\leqslant \|v\|_1 + \|q\|_0 + \beta^{-1} \sup_{(v_h, q_h) \in (X_h, M_h)} \frac{\mathscr{B}((R_h(v, q), Q_h(v, q)); (v_h, q_h))}{\|v_h\|_1 + \|q_h\|_0}$$
$$\leqslant \|v\|_1 + \|q\|_0 + \beta^{-1} \sup_{(v_h, q_h) \in (X_h, M_h)} \frac{B((v, q); (v_h, q_h))}{\|v_h\|_1 + \|q_h\|_0}$$
$$\leqslant c(\|v\|_1 + \|q\|_0). \tag{4.4}$$

Then we see from the definition of (R_h, Q_h), the triangle inequality, and (3.10)–(3.12) that

$$\begin{aligned}
&\|v-R_h(v,q)\|_1+\|q-Q_h(v,q)\|_0\\
&\quad\leqslant\|v-I_hv\|_1+\|q-\rho_hq\|_0+\|I_hv-R_h(v,q)\|_1\\
&\qquad+\|\rho_hq-Q_h(v,q)\|_0\leqslant\|v-I_hv\|_1+\|q-\rho_hq\|_0\\
&\qquad+\beta^{-1}\sup_{(v_h,q_h)\in(X_h,M_h)}\frac{|\mathscr{B}((I_hv-R_h(v,q),\rho_hq-Q_h(v,q));(v_h,q_h))|}{\|v_h\|_1+\|q_h\|_0}\\
&\quad\leqslant\|v-I_hv\|_1+\|q-\rho_hq\|_0\\
&\qquad+\beta^{-1}\sup_{(v_h,q_h)\in(X_h,M_h)}\frac{|\mathscr{B}((I_hv-v,\rho_hq-q);(v_h,q_h))|+|G(q,q_h)|}{\|v_h\|_1+\|q_h\|_0}\\
&\quad\leqslant c(\|v-I_hv\|_1+\|q-\rho_hq\|_0)\\
&\qquad+\beta^{-1}\sup_{(v_h,q_h)\in(X_h,M_h)}\frac{|G(q,q_h)|}{\|v_h\|_1+\|q_h\|_0}\leqslant ch(\|v\|_2+\|q\|_1).
\end{aligned}\tag{4.5}$$

To derive the estimate in the L^2-norm, we consider the dual linearized problem for $(\Phi,\Psi)\in X\times M$ satisfying

$$B((w,r);(\Phi,\Psi))=(w,v-R_h(v,q))\quad\forall(w,r)\in X\times M,\tag{4.6}$$

which satisfies

$$\|\Phi\|_2+\|\Psi\|_1\leqslant c\|v-R_h(v,q)\|_0.\tag{4.7}$$

Obviously, using (3.2) and (4.5) and setting $(w,r)=(e,\eta)=(v-R_h(v,q),q-Q_h(v,q))$ in (4.6) and $(v_h,q_h)=(I_h\Phi,\rho_h\Psi)$ in (4.1), respectively, we see that

$$\begin{aligned}
\|e\|_0^2&=\mathscr{B}((e,\eta);(\Phi-I_h\Phi,\Psi-\rho_h\Psi))\\
&\quad+G(q,\rho_h\Psi)-G(\eta,\Psi)\\
&\leqslant c(\|e\|_1+\|\eta\|_0)(\|\Phi-I_h\Phi\|_1+\|\Psi-\rho_h\Psi\|_0)\\
&\quad+G(q,\rho_h\Psi-\Psi)+G(q,\Psi)-G(\eta,\Psi)\\
&\leqslant ch\{(\|e\|_1+\|\eta\|_0)(\|\Phi\|_2+\|\Psi\|_1)+h\|q\|_1\|\Psi\|_1\}\\
&\leqslant ch(\|e\|_1+\|\eta\|_0+h\|q\|_1)(\|\Phi\|_2+\|\Psi\|_1).
\end{aligned}\tag{4.8}$$

Thus, by combining (4.8) with (4.7) and using (4.5), we deduce

$$\|v-R_h(v,q)\|_0\leqslant ch^2(\|v\|_2+\|q\|_1),$$

which, together with (4.5), yields (4.3). □

Due to $u_0\in D(A)$, we can define $p_0\in H^1(\Omega)\cap M$ (see [17]). Now, we define $(u_{0h},p_{0h})=(R_h(u_0,p_0),Q_h(u_0,p_0))$.

Lemma 4.2. *Under the assumptions of Lemma 2.2 and Theorem 3.1, it holds that, for* $t\in[0,T]$,

$$\|u_h(t)\|_0^2+\int_0^t\Big(\nu\|u_h\|_1^2+G(p_h,p_h)\Big)ds\leqslant c,\tag{4.9}$$

$$\nu\|u_h(t)\|_1^2+G(p_h(t),p_h(t))+\int_0^t\|u_{ht}\|_0^2ds\leqslant c,\tag{4.10}$$

$$\|u(t)-u_h(t)\|_0^2+\int_0^t\Big(\nu\|u-u_h\|_1^2+G(p-p_h,p-p_h)\Big)ds\leqslant ch^2.\tag{4.11}$$

Proof. Taking $(v,q)=2(u_h,p_h)$ in (3.8) and using (2.7) and the definition of $\mathscr{B}(\cdot;\cdot)$, we have

$$\frac{d}{dt}\|u_h\|_0^2+2\nu\|u_h\|_1^2+2G(p_h,p_h)\leqslant\nu\|u_h\|_1^2+\nu^{-1}\gamma^2\|f\|_0^2.$$

Integrating the above inequality from 0 to t and noting

$$\|u_h(0)\|_0\leqslant\|u_0\|_0+\|u_0-R_h(u_0,p_0)\|_0\leqslant c(\|u_0\|_1+\|p_0\|_0),$$

we obtain (4.9).

Subtracting (3.8) from (2.10) with $(v,q)=(v_h,q_h)$, we have

$$\begin{aligned}
&(u_t-u_{ht},v_h)+\mathscr{B}((u-u_h,p-p_h);(v_h,q_h))+b(E+e_h,u,v_h)\\
&\quad+b(u_h,E+e_h,v_h)=G(p,q_h),
\end{aligned}\tag{4.12}$$

for all $(v_h,q_h)\in(X_h,M_h)$, where $(e_h,\eta_h)=(R_h(u,p)-u_h,Q_h(u,p)-p_h)$ and $E=u-R_h(u,p)$. Setting $(v,q)=2(e_h,\eta_h)$ in (4.12) and using (4.1) and (2.7), we deduce

$$\begin{aligned}
&\frac{d}{dt}\|u-u_h\|_0^2+2\nu\|e_h\|_1^2+2G(\eta_h,\eta_h)+2b(E+e_h,u,e_h)\\
&\quad+2b(u_h,E,e_h)=2(u_t-u_{ht},E).
\end{aligned}\tag{4.13}$$

Using Lemma (2.2), (2.3), (2.7), (2.8), and the Young inequality, we see that

$$\begin{aligned}
&|b(E,u,e_h)|\leqslant c\|E\|_1\|u\|_1\|e_h\|_1\leqslant c\|E\|_1^2\|u\|_1^2+\frac{\nu}{8}\|e_h\|_1^2,\\
&|b(e_h,u,e_h)|\leqslant c\Big\{\|e_h\|_0\|e_h\|_1\|u\|_1+\|e_h\|_0^{1/2}\|e_h\|_1^{3/2}\|u\|_0^{1/2}\|u\|_1^{1/2}\Big\}\\
&\qquad\leqslant\frac{\nu}{8}\|e_h\|_1^2+c\Big(1+\|u\|_0^2\Big)\|u\|_1^2\|e_h\|_0^2,\\
&|b(u_h,E,e_h)|\leqslant c\|u_h\|_1\|E\|_1\|e_h\|_1\leqslant\frac{\nu}{8}\|e_h\|_1^2+c\|u_h\|_1^2\|E\|_1^2,\\
&|(u_t-u_{ht},E)|\leqslant c\|E\|_0\|u_t-u_{ht}\|_0,\\
&\|e_h\|_0\leqslant\|u-u_h\|_0+c\|E\|_1.
\end{aligned}$$

Now, combining these inequalities with (4.13), one can find that

$$\begin{aligned}
&\frac{d}{dt}\|u-u_h\|_0^2+\nu\|e_h\|_1^2+G(\eta_h,\eta_h)\\
&\quad\leqslant c\Big(1+\|u\|_0^2\Big)\|u\|_1^2\|u-u_h\|_0^2\\
&\qquad+c\Big(\|u_h\|_1^2+(1+\|u\|_0^2)\|u\|_1^2\Big)\|E\|_1^2\\
&\qquad+c\|E\|_0\|u_t-u_{ht}\|_0.
\end{aligned}\tag{4.14}$$

Then, by integrating (4.14) from 0 to t and noting that

$$\begin{aligned}
&\|u_0-R_h(u_0,p_0)\|_0\leqslant ch^2(\|u_0\|_2+\|p_0\|_1),\\
&\|E\|_0+h\|E\|_1\leqslant ch^2(\|u\|_2+\|p\|_1),
\end{aligned}\tag{4.15}$$

it follows from the Schwarz inequality, Lemma 2.2 and (4.9) that

$$\|u(t)-u_h(t)\|_0^2+\int_0^t\left(\nu\|e_h\|_1^2+G(\eta_h,\eta_h)\right)\mathrm{d}s$$
$$\leqslant ch^4+c\int_0^t\|u-u_h\|_0^2\mathrm{d}s$$
$$+ch^2\left(\int_0^t\left(\|u\|_2^2+\|p\|_1^2\right)\mathrm{d}s\right)^{1/2}\left(\int_0^t\left(\|u_t\|_0^2+\|u_{ht}\|_0^2\right)\mathrm{d}s\right)^{1/2}$$
$$+ch^2\int_0^t\left(\|u_h\|_1^2+\left(1+\|u\|_0^2\right)\|u\|_1^2\right)\left(\|u\|_2^2+\|p\|_1^2\right)\mathrm{d}s\Big\}$$
$$\leqslant ch^2+ch^2\left(1+\int_0^t\|u_{ht}\|_0^2\mathrm{d}s\right)^{1/2}+c\int_0^t\|u-u_h\|_0^2\mathrm{d}s. \tag{4.16}$$

Applying Lemma 2.1 to (4.16) gives

$$\|u(t)-u_h(t)\|_0^2+\int_0^t\left(\nu\|e_h\|_1^2+G(\eta_h,\eta_h)\right)\mathrm{d}s$$
$$\leqslant ch^2\left\{1+\left(\int_0^t\|u_{ht}\|_0^2\mathrm{d}s\right)^{1/2}\right\}, \tag{4.17}$$

which, together with (3.12), (4.3) and Lemma 2.2, yields

$$\|u(t)-u_h(t)\|_0^2+\int_0^t\left(\nu\|u-u_h\|_1^2+G(p-p_h,p-p_h)\right)\mathrm{d}s$$
$$\leqslant ch^2\left\{1+\left(\int_0^t\|u_{ht}\|_0^2\mathrm{d}s\right)^{1/2}\right\}. \tag{4.18}$$

To estimate $\int_0^t\|u_{ht}\|_0^2\mathrm{d}s$, we differentiate the term $d(u_h,q_h)+G(p_h,q_h)$ with respect to time t in (3.8) and set $(v_h,q_h)=(u_{ht},p_h)$ to have

$$\|u_{ht}\|_0^2+\frac{1}{2}\frac{d}{\mathrm{d}t}\left(\nu\|u_h\|_1^2+G(p_h,p_h)\right)+b(u_h,u,u_{ht})$$
$$+b(u_h,u-u_h,u_{ht})=(f,u_{ht}). \tag{4.19}$$

Thanks to (2.8), (2.9) and (3.3), the trilinear terms can be bounded as follows:

$$|b(u_h,u,u_{ht})|\leqslant c\|u_h\|_1\|u\|_2\|u_{ht}\|_0$$
$$\leqslant\frac{1}{8}\|u_{ht}\|_0^2+c\|u\|_2^2\|u_h\|_1^2,\ |b(u_h,u-u_h,u_{ht})|$$
$$\leqslant c\Big\{\|u_h\|_1^{1/2}\|u_h\|_0^{1/2}\|u_{ht}\|_1^{1/2}\|u_{ht}\|_0^{1/2}\|u-u_h\|_1$$
$$+\|u_h\|_1^{1/2}\|u_h\|_0^{1/2}\|u-u_h\|_1^{1/2}\|u-u_h\|_0^{1/2}\|u_{ht}\|_1\Big\}$$
$$\leqslant\frac{1}{8}\|u_{ht}\|_0^2+ch^{-2}\left(\|u_h\|_0^2\|u-u_h\|_1^2+\|u_h\|_1^2\|u-u_h\|_0^2\right).$$

Subsequently, combining the above estimates with (4.19) gives

$$\|u_{ht}\|_0^2+\frac{d}{\mathrm{d}t}\left(\nu\|u_h\|_1^2+G(p_h,p_h)\right)$$
$$\leqslant c\|u\|_2^2\|u_h\|_1^2+ch^{-2}\left(\|u_h\|_0^2\|u-u_h\|_1^2+\|u_h\|_1^2\|u-u_h\|_0^2\right). \tag{4.20}$$

Now, integrating (4.20) from 0 to t, using Lemma 2.2, (4.9), and noting

$$\nu\|u_{0h}\|_1^2+G(p_{0h},p_{0h})\leqslant c\left(\|u_{0h}\|_1^2+\|p_{0h}\|_0^2\right)$$
$$\leqslant c\left(\|u_0\|_1^2+\|p_0\|_0^2\right),$$

we obtain

$$\int_0^t\|u_{ht}\|_0^2\mathrm{d}s+\nu\|u_h(t)\|_1^2+G(p_h(t),p_h(t))$$
$$\leqslant\nu\|u_{0h}\|_1^2+G(p_{0h},p_{0h})+\int_0^t\|u\|_2^2\|u_h\|_1^2\mathrm{d}s$$
$$+ch^{-2}\int_0^t\left(\|u_h\|_0^2\|u-u_h\|_1^2+\|u_h\|_1^2\|u-u_h\|_0^2\right)\mathrm{d}s$$
$$\leqslant c+c\left(\int_0^t\|u_{ht}\|_0^2\mathrm{d}s\right)^{1/2}. \tag{4.21}$$

Combining (4.21) with (4.18) gives (4.10) and (4.11). □

Lemma 4.3. *Under the assumptions of Lemma 2.2 and Theorem 3.1, it holds that, for* $t\in[0,T]$,

$$\nu\tau(t)\|u(t)-u_h(t)\|_1^2+\int_0^t\tau(s)\|u_t-u_{ht}\|_0^2\mathrm{d}s\leqslant ch^2. \tag{4.22}$$

Proof. Differentiating the term $d(u-u_h,q_h)+G(p-p_h,q_h)$ in (4.12), taking $(v_h,q_h)=(e_{ht},\eta_h)$ in (4.12) and using (4.1), we see that

$$\frac{1}{2}\|e_{ht}\|_0^2+\frac{1}{2}\|u_t-u_{ht}\|_0^2+\frac{1}{2}\frac{d}{\mathrm{d}t}(\nu\|e_h\|_1^2+G(\eta_h,\eta_h))$$
$$\leqslant|b(u-u_h,u,e_{ht})|+|b(u,u-u_h,e_{ht})|$$
$$+|b(u-u_h,u-u_h,e_{ht})|+\frac{1}{2}\|E_t\|_0^2. \tag{4.23}$$

Due to (2.7), (2.8), (2.9) and Lemma 2.2, we have

$$|b(u-u_h,u,e_{ht})+b(u,u-u_h,e_{ht})|$$
$$\leqslant c\|u\|_2\|u-u_h\|_1\|e_{ht}\|_0$$
$$\leqslant\frac{1}{8}\|e_{ht}\|_0^2+c\|u\|_2^2\|u-u_h\|_1^2,$$
$$|b(u-u_h,u-u_h,e_{ht})|$$
$$\leqslant c\|u-u_h\|_0^{1/2}\|u-u_h\|_1^{3/2}\|e_{ht}\|_0^{1/2}\|e_{ht}\|_1^{1/2}$$
$$\leqslant\frac{1}{8}\|e_{ht}\|_0^2+ch^{-1}\|u-u_h\|_0\|u-u_h\|_1^3.$$

Hence, combining these inequalities with (4.23) yields

$$\|u_t-u_{ht}\|_0^2+\frac{d}{\mathrm{d}t}\left(\nu\|e_h\|_1^2+G(\eta_h,\eta_h)\right)$$
$$\leqslant c\|E_t\|_0^2+c\|u\|_2^2\|u-u_h\|_1^2+ch^{-1}\|u-u_h\|_0\|u-u_h\|_1^3. \tag{4.24}$$

Multiplying (4.24) by $\tau(t)$, integrating from 0 to t, and using Lemmas 2.2, 4.1 and 4.2, we deduce

$$\int_0^t\tau(s)\|u_t-u_{ht}\|_0^2\mathrm{d}s+\tau(t)\left(\nu\|e_h(t)\|_1^2+G(\eta_h(t),\eta_h(t))\right)$$
$$\leqslant c\int_0^t\left(\nu\|e_h\|_1^2+G(\eta_h,\eta_h)\right)\mathrm{d}s$$
$$+c\int_0^t\tau(s)\|E_t\|_0^2\mathrm{d}s+ch^2\leqslant ch^2,$$

which, together with Lemmas 2.2, 4.1 and 4.2, yields (4.22). □

Lemma 4.4. *Under the assumptions of Lemma 2.2 and Theorem 3.1, it holds that, for* $t \in [0,T]$,

$$\|u_{ht}(t)\|_0^2 + \int_0^t \Big(\nu\|u_{ht}\|_1^2 + G(p_{ht},p_{ht})\Big)\,\mathrm{d}s \leqslant c, \tag{4.25}$$

$$\tau(t)\Big(\nu\|u_{ht}(t)\|_1^2 + G(p_{ht}(t),p_{ht}(t))\Big) + \int_0^t \tau(s)\|u_{htt}\|_0^2\,\mathrm{d}s \leqslant ch^2, \tag{4.26}$$

$$\tau(t)\|u_t(t)-u_{ht}(t)\|_0^2 + \int_0^t \tau(s)\Big(\nu\|e_{ht}\|_1^2 + G(\eta_{ht},\eta_{ht})\Big)\,\mathrm{d}s \leqslant ch^2. \tag{4.27}$$

Proof. By differentiating (3.8) with respect to time, it follows that

$$(u_{htt},v_h) + \mathscr{B}((u_{ht},p_{ht});(v_h,q_h)) + b(u_{ht},u_h,v_h) + b(u_h,u_{ht},v_h) = (f_t,v_h), \tag{4.28}$$

for all $(v_h,q_h) \in (X_h,M_h)$. Taking $(v_h,q_h) = 2(u_{ht},p_{ht})$ in (4.28) and using (2.7), we deduce

$$\frac{d}{\mathrm{d}t}\|u_{ht}\|_0^2 + 2\nu\|u_{ht}\|_1^2 + 2G(p_{ht},p_{ht}) + 2b(u_{ht},u_h,u_{ht}) \leqslant \frac{\nu}{4}\|u_{ht}\|_1^2 + c\|f_t\|_0^2. \tag{4.29}$$

Using (2.7) and (2.8), we have

$$2|b(u_{ht},u_h,u_{ht})| \leqslant c\|u_{ht}\|_0^{1/2}\|u_{ht}\|_1^{3/2}\|u_h\|_0^{1/2}\|u_h\|_1^{1/2} \leqslant \frac{\nu}{4}\|u_{ht}\|_1^2 + c\|u_h\|_0^2\|u_h\|_1^2\|u_{ht}\|_0^2,$$

which, combined with (4.29), gives

$$\frac{d}{\mathrm{d}t}\|u_{ht}\|_0^2 + \nu\|u_{ht}\|_1^2 + G(p_{ht},p_{ht}) \leqslant c\|u_h\|_0^2\|u_h\|_1^2\|u_{ht}\|_0^2 + c\|f_t\|_0^2. \tag{4.30}$$

Integrating (4.30) and using Lemma 4.2, we obtain (4.25).

Then, differentiating again the term $d(u_{ht},q_h) + G(p_{ht},q_h)$ in (4.28) and taking $(v_h,q_h) = (u_{htt},p_{ht})$, we see that

$$\|u_{htt}\|_0^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\nu\|u_{ht}\|_1^2 + G(p_{ht},p_{ht})\Big) + b(u,u_{ht},u_{htt}) + b(u_{ht},u,u_{htt}) + b(u_{ht},u_h-u,u_{htt}) + b(u_h-u,u_{ht},u_{htt}) \leqslant \frac{1}{8}\|u_{htt}\|_0^2 + c\|f_t\|_0^2. \tag{4.31}$$

Obviously, it follows from (2.7)–(2.9), (3.3), (4.10) and (4.11) that

$$|b(u,u_{ht},u_{htt}) + b(u_{ht},u,u_{htt})| \leqslant c\|u\|_2\|u_{ht}\|_1\|u_{htt}\|_0 \leqslant \frac{1}{8}\|u_{htt}\|_0^2 + c\|u\|_2^2\|u_{ht}\|_1^2,$$

$$|b(u_{ht},u_h-u,u_{htt}) + b(u_h-u,u_{ht},u_{htt})| \leqslant ch^{-1}\|u_{ht}\|_0^{1/2}\|u_{ht}\|_1^{1/2}\|u-u_h\|_0^{1/2}\|u-u_h\|_1^{1/2}\|u_{htt}\|_0 \leqslant \frac{1}{8}\|u_{htt}\|_0^2 + ch^{-2}\|u_{ht}\|_0\|u_{ht}\|_1\|u-u_h\|_0\|u-u_h\|_1.$$

Then, combining these estimates with (4.31), it follows from Lemma 2.2 that

$$\|u_{htt}\|_0^2 + \frac{d}{\mathrm{d}t}\Big(\nu\|u_{ht}\|_1^2 + G(p_{ht},p_{ht})\Big) \leqslant c\|u\|_2^2\|u_{ht}\|_1^2 + ch^{-2}\|u_{ht}\|_0\|u_{ht}\|_1\|u - u_h\|_0\|u-u_h\|_1 + c\|f_t\|_0^2. \tag{4.32}$$

Similarly, multiplying (4.32) by $\tau(t)$, integrating from 0 to t, and using Lemma 4.2 and (A2), we see that

$$\int_0^t \tau(s)\|u_{htt}\|_0^2\,\mathrm{d}s + \tau(t)\Big(\nu\|u_{ht}\|_1^2 + G(p_{ht},p_{ht})\Big) \leqslant c\int_0^t \Big(\nu\|u_{ht}\|_1^2 + G(p_{ht},p_{ht})\Big)\,\mathrm{d}s + c\int_0^t \|f_t\|_0^2\,\mathrm{d}s + c\int_0^t \Big(\|u\|_2^2\|u_{ht}\|_1^2 + h^{-2}\|u-u_h\|_0\|u-u_h\|_1\|u_{ht}\|_0\|u_{ht}\|_1\Big)\,\mathrm{d}s. \tag{4.33}$$

Combining (4.33) with and (4.25), (4.11) and using Lemma 2.2 completes the proof of (4.26).

To show (4.27), differentiating (4.12) with respect to time t and using (4.1) gives

$$(u_{tt}-u_{htt},v_h) + \mathscr{B}((e_{ht},\eta_{ht});(v_h,q_h)) + b(u_t-u_{ht},u,v_h) + b(u-u_h,u_t,v_h) + b(u_t,u-u_h,v_h) + b(u,u_t-u_{ht},v_h) - b(u_t-u_{ht},u-u_h,v_h) - b(u-u_h,u_t-u_{ht},v_h) = 0, \tag{4.34}$$

for all $(v_h,q_h) \in (X_h,M_h)$. Taking $(v_h,q_h) = (e_{ht},\eta_{ht})$ in (4.34), we see that

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\|u_t-u_{ht}\|_0^2 + \nu\|e_{ht}\|_1^2 + G(\eta_{ht},\eta_{ht}) + b(u_t-u_{ht},u,e_{ht}) + b(u-u_h,u_t,e_{ht}) + b(u_t,u-u_h,e_{ht}) + b(u,u_t-u_{ht},e_{ht}) - b(u_t-u_{ht},u-u_h,e_{ht}) - b(u-u_h,u_t-u_{ht},e_{ht}) = (u_{tt}-u_{htt},E_t). \tag{4.35}$$

Thanks to 2.3 and (2.7)–(2.9), we estimate the trilinear terms in (4.35) as follows:

$$|b(u_t-u_{ht},u,e_{ht}) + b(u,u_t-u_{ht},e_{ht})| \leqslant c\|u\|_2\|e_{ht}\|_1\|u_t-u_{ht}\|_0 \leqslant \frac{\nu}{8}\|e_{ht}\|_1^2 + c\|u\|_2^2\|u_t-u_{ht}\|_0^2,$$

$$|b(u-u_h,u_t,e_{ht})| + |b(u_t,u-u_h,e_{ht})| \leqslant c\|u_t\|_1\|e_{ht}\|_1\|u-u_h\|_1 \leqslant \frac{\nu}{8}\|e_{ht}\|_1^2 + c\|u_t\|_1^2\|u-u_h\|_1^2,$$

$$|b(u_t-u_{ht},u-u_h,e_{ht})|+|b(u-u_h,u_t-u_{ht},e_{ht})|$$
$$\leqslant c\|u_t-u_{ht}\|_1\|u-u_h\|_1\|e_{ht}\|_1$$
$$\leqslant \frac{\nu}{8}\|e_{ht}\|_1^2+c(\|u_t\|_1^2+\|u_{ht}\|_1^2)\|u-u_h\|_1^2,$$
$$|(u_{tt}-u_{htt},E_t)|\leqslant \|u_{tt}-u_{htt}\|_0\|E_t\|_0$$
$$\leqslant ch^2(\|u_{tt}\|_0+\|u_{htt}\|_0)(\|u_t\|_2+\|p_t\|_1).$$

Combining these inequalities with (4.35) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_t-u_{ht}\|_0^2+\nu\|e_{ht}\|_1^2+G(\eta_{ht},\eta_{ht})$$
$$\leqslant c\|u\|_2^2\|u_t-u_{ht}\|_0^2+c\Big(\|u_t\|_1^2+\|u_{ht}\|_1^2\Big)\|u-u_h\|_1^2$$
$$+ch^2(\|u_{tt}\|_0+\|u_{htt}\|_0)(\|u_t\|_2+\|p_t\|_1). \tag{4.36}$$

Multiplying (4.36) by $\tau(t)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\tau(t)\|u_t-u_{ht}\|_0^2\Big)+\tau(t)\Big(\nu\|e_{ht}\|_1^2+G(\eta_{ht},\eta_{ht})\Big)$$
$$\leqslant c\Big(1+\|u\|_2^2\Big)\tau(t)\|u_t-u_{ht}\|_0^2$$
$$+c\Big(\|u_t\|_1^2+\|u_{ht}\|_1^2\Big)\tau(t)\|u-u_h\|_1^2$$
$$+ch^2\tau(t)(\|u_{tt}\|_0+\|u_{htt}\|_0)(\|u_t\|_2+\|p_t\|_1). \tag{4.37}$$

Finally, we integrate (4.37) from 0 to t and apply Lemmas 2.2, 4.2 and 4.3 to obtain

$$\tau(t)\|u_t(t)-u_{ht}(t)\|_0^2+\int_0^t\tau(s)\Big(\nu\|e_{ht}\|_1^2+G(\eta_{ht},\eta_{ht})\Big)\,\mathrm{d}s$$
$$\leqslant c\int_0^t\Big(1+\|u\|_2^2\Big)\tau(s)\|u_t-u_{ht}\|_0^2\mathrm{d}s$$
$$+c\int_0^t\Big(\|u_t\|_1^2+\|u_{ht}\|_1^2\Big)\tau(s)\|u-u_h\|_1^2\,\mathrm{d}sch^2\int_0^t\tau(s)(\|u_{tt}\|_0$$
$$+\|u_{htt}\|_0)(\|u_t\|_2+\|p_t\|_1)\,\mathrm{d}s\leqslant ch^2,$$

which completes the proof. □

Lemma 4.5. *Under the assumptions of Lemma 2.2 and Theorem 3.1, it holds that, for $t\in[0,T]$,*

$$\tau^{1/2}(t)\|p(t)-p_h(t)\|_0\leqslant ch. \tag{4.38}$$

Proof. It follows from the *inf-sup* condition (3.11) and (4.12) that

$$\|\eta_h(t)\|_0\leqslant\beta^{-1}\sup_{(v_h,q_h)\in(X_h,M_h)}\frac{\mathscr{B}((e_h(t),\eta_h(t));(v_h,q_h))}{\|v_h\|_1+\|q_h\|_0}$$
$$\leqslant\beta^{-1}\gamma\|u_t(t)-u_{ht}(t)\|_0+c(\|u(t)\|_1$$
$$+\|u_h(t)\|_1)\|u(t)-u_h(t)\|_1. \tag{4.39}$$

Using Lemmas 4.1–4.4 we see that

$$\tau^{1/2}(t)\|\eta_h(t)\|_0\leqslant c\tau^{1/2}(t)\|u_t(t)-u_{ht}(t)\|_0+c(\|u(t)\|_1$$
$$+\|u_h(t)\|_1)\tau^{1/2}(t)\|u(t)-u_h(t)\|_1\leqslant ch. \tag{4.40}$$

Thus, using Lemmas 2.2 and 4.1 again yields

$$\tau^{1/2}(t)\|p(t)-p_h(t)\|_0$$
$$\leqslant\tau^{1/2}(t)\|\eta_h(t)\|_0+\tau^{1/2}(t)\|p(t)-Q_h(u(t),p(t))\|_0$$
$$\leqslant ch+ch(\|u(t)\|_2+\|p(t)\|_1)\leqslant ch,$$

which is (4.38). □

Theorem 4.6. *Under the assumptions of Lemma 2.2 and Theorem 3.1, it holds that, for $t\in[0,T]$,*

$$\tau^{1/2}(t)\|u(t)-u_h(t)\|_1+\tau^{1/2}(t)\|p(t)-p_h(t)\|_0\leqslant ch. \tag{4.41}$$

This theorem follows from Lemmas 4.3 and 4.5.

5. L^2-Error estimates

In this section we estimate the error $\|u-u_h\|_0$ using a parabolic duality argument for a backward-in-time linearized Navier–Stokes problem [17,18]. The dual problem is to seek $(\Phi(t),\Psi(t))\in X\times M$ such that, for $t\in[0,T]$ and $g\in L^2(0,T,Y)$,

$$(v,\Phi_t)-B((v,q);(\Phi,\Psi))-b(u,v,\Phi)-b(v,u,\Phi)=(v,g), \tag{5.1}$$

for all $(v,q)\in(X,M)$, with $\Phi(T)=0$. This problem is well-posed and has a unique solution (Φ,Ψ) with [18]

$$\Phi\in C(0,T,V)\cap L^2(0,T,D(A))\cap H^1(0,T,Y),$$
$$\Psi\in L^2(0,T,H^1(\Omega)\cap M).$$

First, we need to recall the following regularity results provided by Hill and Süli in [18].

Lemma 5.1. *The solution (Φ,Ψ) of (5.1) satisfies*

$$\sup_{0\leqslant t\leqslant T}\|\Phi(t)\|_1^2+\int_0^T\Big(\|\Phi\|_2^2+\|\Psi\|_1^2+\|\Phi_t\|_0^2\Big)\mathrm{d}t\leqslant c\int_0^T\|g\|_0^2\mathrm{d}t. \tag{5.2}$$

Lemma 5.2. *Under the assumptions of Lemma 2.2 and Theorem 3.1, it holds that*

$$\int_0^T\|u-u_h\|_0^2\,\mathrm{d}s\leqslant ch^4. \tag{5.3}$$

Proof. Here we introduce the dual Galerkin projection $(\Phi_h(t),\Psi_h(t))$ of $(\Phi(t),\Psi(t))$ such that

$$\mathscr{B}((v_h,q_h));(\Phi_h,\Psi_h)=B((v_h,q_h);(\Phi,\Psi))\quad\forall(v_h,q_h)\in(X_h,M_h),$$

which yields

$$\mathscr{B}((v_h,q_h);(\Phi-\Phi_h,\Psi-\Psi_h))=G(q_h,\Psi)\quad\forall(v_h,q_h)\in(X_h,M_h). \tag{5.4}$$

By using a similar approach to the proof of Lemma 4.1, we can prove

$$\|\Phi-\Phi_h\|_0+h\|\Phi-\Phi_h\|_1+h\|\Psi-\Psi_h\|_0 \leqslant ch^2(\|\Phi\|_2+\|\Psi\|_1). \tag{5.5}$$

Taking $(v_h, q_h)=(\Phi_h,\Psi_h)$ in (4.12), we have

$$(e_t,\Phi_h)+\mathscr{B}((e,\eta);(\Phi_h,\Psi_h))+b(u,e,\Phi_h) + b(e,u,\Phi_h)-b(e,e,\Phi_h)=G(\eta,\Psi_h), \tag{5.6}$$

where $(e,\eta)=(u-u_h,p-p_h)$. Adding (5.6) and (5.1) with $(v,q)=(e,\eta)$ and $g=e$, we see that

$$\|e\|_0^2=\frac{d}{dt}(e,\Phi)-(e_t,\Phi-\Phi_h)-\mathscr{B}((e,\eta); (\Phi-\Phi_h,\Psi-\Psi_h))-b(e,u,\Phi-\Phi_h)-b(u,e,\Phi-\Phi_h) - b(e,e,\Phi_h)+G(\eta,\Psi-\Psi_h). \tag{5.7}$$

Applying (2.7), (2.8), (2.9) and Lemma 5.1, we have

$$|(e_t,\Phi-\Phi_h)|\leqslant c(\|u_t\|_0+\|u_{ht}\|_0)\|\Phi-\Phi_h\|_0 \leqslant ch^2(\|u_t\|_0+\|u_{ht}\|_0)(\|\Phi\|_2+\|\Psi\|_1),$$
$$|b(e,u,\Phi-\Phi_h)+b(u,e,\Phi-\Phi_h)|\leqslant c\|u\|_1\|e\|_1\|\Phi-\Phi_h\|_1 \leqslant ch\|u\|_1\|e\|_1\|\Phi\|_2,$$

$$|b(e,e,\Phi_h)|\leqslant c\|e\|_1^2\|\Phi_h\|_1\leqslant c\|e\|_1^2(h\|\Phi\|_2+\|\Phi\|_1),$$
$$|G(\eta,\Psi-\Psi_h)|\leqslant chG^{1/2}(\eta,\eta)\|\Psi\|_1.$$

As for the bilinear term, by using (5.4), (4.3), (3.6) and (3.12), we have the following estimate

$$\begin{aligned}&|\mathscr{B}((e,\eta);(\Phi-\Phi_h,\Psi-\Psi_h))|\\ &\quad\leqslant|\mathscr{B}((u-R_h(u,p),p-Q_h(u,p)); (\Phi-\Phi_h,\Psi-\Psi_h))|+|G(Q_h(u,p)-p_h,\Psi)|\\ &\quad\leqslant c(\|u-R_h(u,p)\|_1+\|p-Q_h(u,p)\|_0)(\|\Phi-\Phi_h\|_1 +\|\Psi-\Psi_h\|_0)+|G(Q_h(u,p)-p+\eta,\Psi)|\\ &\quad\leqslant ch^2(\|u\|_2+\|p\|_1)(\|\Phi\|_2+\|\Psi\|_1)+chG^{1/2}(\eta,\eta)\|\Psi\|_1.\end{aligned}$$

Then, combining the above estimates with (5.4), we see that

$$\begin{aligned}\|e\|_0^2=&\frac{d}{dt}(e,\Phi)+ch(h\|u_t\|_0+h\|u_{ht}\|_0+\|u\|_1\|e\|_1 \\ &+G^{1/2}(\eta,\eta))(\|\Phi\|_2+\|\Psi\|_1)\\ &+ch^2(\|u\|_2+\|p\|_1)(\|\Phi\|_2+\|\Psi\|_1)\\ &+c\|e\|_1^2(h\|\Phi\|_2+\|\Phi\|_1).\end{aligned} \tag{5.8}$$

Integrating (5.8) from 0 to T yields

$$\begin{aligned}\int_0^T\|e(s)\|_0^2\,ds=&-(e(0),\Phi(0))\\ &+ch\left(\int_0^T\left(h^2\|u_t\|_0^2+h^2\|u_{ht}\|_1^2+\|u\|_1^2\|e\|_1^2 + G(\eta,\eta)\right)ds\right)^{1/2}\left(\int_0^T\left(\|\Phi\|_2^2+\|\Psi\|_1^2\right)ds\right)^{1/2}\\ &+ch^2\left(\int_0^t\left(\|u\|_2^2+\|p\|_1^2\right)ds\right)^{1/2} \times\left(\int_0^t\left(\|\Phi\|_2^2+\|\Psi\|_1^2\right)ds\right)^{1/2}\\ &+c\int_0^T\|e\|_1^2(h\|\Phi\|_2+\|\Phi\|_1)\,ds.\end{aligned} \tag{5.9}$$

In addition, by the definition of R_h, we have

$$|(e(0),\Phi(0))|=|(u_0-R_h(u_0,p_0),\Phi(0))| \leqslant ch^2(\|u_0\|_2+\|p_0\|_1)\|\Phi(0)\|_1. \tag{5.10}$$

Combining (5.10) with (5.9) and using (5.2) with $g=e$ completes the proof of (5.3). □

Lemma 5.3. *Under the assumptions of Lemmas 2.2 and 5.1 and Theorem 3.1, it holds that, for* $t\in[0,T]$,

$$\tau^{1/2}(t)\|u(t)-u_h(t)\|_0\leqslant ch^2. \tag{5.11}$$

Proof. Taking $(v_h,q_h)=2(e_h,\eta_h)=2(R_h(u,p)-u_h, Q_h(u,p)-p_h)$ in (4.12) and using (4.1), we see that

$$\frac{d}{dt}\|e_h\|_0^2+2\nu\|e_h\|_1^2+2G(\eta_h,\eta_h)+2b(u,u-u_h,e_h) + 2b(u-u_h,u,e_h)-2b(u-u_h,u-u_h,e_h)\leqslant 2\|E_t\|_0\|e_h\|_0. \tag{5.12}$$

Obviously, using (2.7), (2.8), and setting $E=u-R_h(u,p)$, we have

$$2|b(u,u-u_h,e_h)+b(u-u_h,u,e_h)|\leqslant c\|u\|_2\|e_h\|_1\|u-u_h\|_0 \leqslant\frac{\nu}{4}\|e_h\|_1^2+c\|u\|_2^2\|u-u_h\|_0^2, 2|b(u-u_h,u-u_h,e_h)| \leqslant\|u-u_h\|_1^2\|e_h\|_1\leqslant\frac{\nu}{4}\|e_h\|_1^2+c\|u-u_h\|_1^4.$$

Thus, combining these estimates with (5.12) gives

$$\frac{d}{dt}\|e_h\|_0^2+\nu\|e_h\|_1^2+G(\eta_h,\eta_h) \leqslant 2\|E_t\|_0\|e_h\|_0+c\|u\|_2^2\|u-u_h\|_0^2+c\|u-u_h\|_1^4. \tag{5.13}$$

Using Lemmas 5.2, 2.2 and (4.3), we have

$$\begin{aligned}\int_0^T\|e_h\|_0^2\,ds&\leqslant 2\int_0^T\|u-u_h\|_0^2\,ds+2\int_0^T\|E\|_0^2\,ds\\ &\leqslant 2\int_0^T\|u-u_h\|_0^2\,ds + ch^4\int_0^T\left(\|u\|_2^2+\|p\|_1^2\right)ds\leqslant ch^4.\end{aligned} \tag{5.14}$$

Multiplying (5.13) by $\tau(t)$ and integrating from 0 to t, and using Lemmas 4.2, 4.3, 5.2 and (5.14), we obtain

$$\tau(t)\|e_h(t)\|_0^2 + \int_0^t \tau(s)\left(v\|e_h\|_1^2 + G(\eta_h,\eta_h)\right)\mathrm{d}s \leqslant c\int_0^t \|e_h\|_0^2\,\mathrm{d}s + c\int_0^t \|u\|_2^2\|u-u_h\|_0^2\,\mathrm{d}s + c\int_0^t \tau(s)\|u-u_h\|_1^4\,\mathrm{d}s + c\left(\int_0^t \tau(s)\|E_t\|_0^2\,\mathrm{d}s\right)^{1/2}\left(\int_0^t \|e_h\|_0^2\,\mathrm{d}s\right)^{1/2} \leqslant ch^4. \tag{5.15}$$

Using again (4.3) and Lemma 2.2, we have

$$\|u(t) - R_h(u(t),p(t))\|_0^2 \leqslant ch^4(\|u(t)\|_2^2 + \|p(t)\|_1^2) \leqslant ch^4. \tag{5.16}$$

Combining (5.15) with (5.16) yields (5.11). □

The next theorem follows from Lemma 5.3 and Theorem 4.6.

Theorem 5.4. *Under the assumptions of Lemma 2.2 and Theorem 3.1, it holds that, for* $t \in [0,T]$,

$$\|u(t)-u_h(t)\|_0 + h\|u(t)-u_h(t)\|_1 + h\|p(t)-p_h(t)\|_0 \leqslant c\tau^{-1/2}(t)h^2. \tag{5.17}$$

Remark. Unfortunately, we can not drop the singular factor $\tau^{-1/2}(t)$ in the above error estimates because of some technique difficulties.

6. Numerical experiments

In this section we present numerical results to compare the new stabilized finite element method presented in Section 3 with other finite element methods for the two-dimensional transient Stokes equations and to illustrate the convergence theory of this new method for the transient Navier–Stokes equations developed in Sections 4 and 5. In all experiments here, we consider the transient Navier–Stokes equations on $\Omega = [0,1]\times[0,1]$ with a body force $f(x,t)$ such that the true solution is

$$u(x,t) = (u_1(x,t), u_2(x,t)), \quad p(x,t) = 10(2x_1-1)(2x_2-1)\cos(t),$$

$$u_1(x,t) = 10x_1^2(x_1-1)^2 x_2(x_2-1)(2x_2-1)\cos(t),$$

$$u_2(x,t) = -10x_1(x_1-1)(2x_1-1)x_2^2(x_2-1)^2\cos(t).$$

The domain is divided into triangles (see Fig. 1). All the numerical experiments have been performed using the conforming P_1 finite element for both velocity and pressure. The implicit (backward) Euler scheme is used for the time discretization at $t = 1$, with the time step $\mathrm{d}t = 0.0025$.

We first compare the new stabilized finite element method with the standard Galerkin method, the penalty

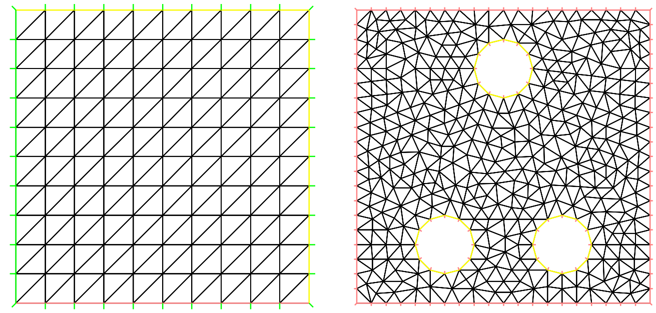


Fig. 1. Uniform and unstructured triangulation of Ω into triangles.

Table 1
Comparison of the results for the transient Navier–Stokes equations with different stabilized parameter δ (GLS method with $v = 0.01$ and $1/h = 27$ on uniform mesh)

δ	$\frac{\|u-u_h\|_0}{\|u\|_0}$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	$\|\mathrm{div}(u-u^h)\|^{0,\infty,K}$
1.25e+010	0.58186	0.780421	0.0879626	0.000429764
1250	0.524077	0.714703	0.00231233	0.000417508
125	0.257037	0.414099	0.00136363	0.00034068
12.5	0.0420664	0.154004	0.0010947	0.000164305
1.2500	0.010635	0.10718	0.00106821	4.23749e−005
0.125	0.00977313	0.104804	0.00106896	2.90663e−005
0.0125	0.0109201	0.108888	0.00116981	2.94516e−005
0.000125	0.0136091	0.126283	0.0020777	3.26249e−005
1.25e−010	0.0137342	0.127326	0.0872114	3.27503e−005

Table 2
Comparison of the results for the transient Navier–Stokes equations with different stabilized parameter δ_1 (multiscale enrichment method with $v = 0.01$, $\delta_2 = 1/(12v)$ and $1/h = 27$ on uniform mesh)

δ_1	$\frac{\|u-u_h\|_0}{\|u\|_0}$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	$\|\mathrm{div}(u-u^h)\|^{0,\infty,K}$
1.25e+010	0.578187	0.764985	0.212368	0.000385109
1250	0.520675	0.699656	0.0023136	0.000374259
125	0.255029	0.401199	0.0013685	0.000306779
12.5	0.0420326	0.147995	0.00109729	0.000152181
1.25	0.012678	0.106257	0.00106908	3.8449e−005
0.125	0.0121566	0.104464	0.00107023	2.84652e−005
0.00125	0.0152158	0.119877	0.00177986	3.12862e−005
0.000125	0.0133481	0.108487	0.00118251	2.91604e−005
1.25e−010	0.0161279	0.127469	0.084931	3.23125e−005

Table 3
The results for the transient Navier–Stokes equations (Galerkin method with $v = 0.01$ on uniform mesh)

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	$\|\mathrm{div}(u-u^h)\|^{0,\infty,K}$
18	0.0308098	0.189134	1.01439	0.000102231
27	0.013734	0.127327	4.06744	3.27466e−005
36	0.00758175	0.0939097	0.444616	1.43189e−005
45	0.00490889	0.0761692	0.151242	7.48351e−006
54	0.00335357	0.0624615	0.12808	4.38911e−006
63	0.00249719	0.0543352	0.514549	2.78988e−006
72	0.00188206	0.0467924	0.461222	1.88213e−006
81	0.00150824	0.0422297	0.110548	1.32893e−006

Table 4
The results for the transient Navier–Stokes equations (GLS method with $\nu = 0.01$ and $\delta = 0.0125\nu$ on uniform mesh)

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	$\|\text{div}(u-u^h)\|_{0,\infty,K}$
18	0.0319185	0.182044	0.00241892	0.000200833
27	0.010635	0.10718	0.00106821	4.23749e–005
36	0.00551077	0.0784519	0.000599516	1.42401e–005
45	0.00343014	0.0623285	0.000383294	6.05239e–006
54	0.00235515	0.0518077	0.00026603	3.37212e–006
63	0.00172107	0.0443561	0.000195388	2.14376e–006
72	0.00131399	0.038789	0.000149563	1.44665e–006
81	0.00103657	0.0344677	0.000118158	1.02167e–006

Table 5
The results for the transient Navier–Stokes equations (multiscale enrich method with $\nu = 0.01$, $\delta_2 = \frac{1}{12\nu}$ and $\delta_1 = 0.0125\nu$ on uniform mesh)

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	$\|\text{div}(u-u^h)\|_{0,\infty,K}$
18	0.0348923	0.176206	0.00242275	0.000194556
27	0.012678	0.106257	0.00106908	3.8449e–005
36	0.00678573	0.0781985	0.000599852	1.21692e–005
45	0.00427598	0.0622291	0.000383471	5.63247e–006
54	0.00295196	0.0517581	0.000266142	3.30058e–006
63	0.00216323	0.044327	0.000195468	2.09918e–006
72	0.00165422	0.0387699	0.000149626	1.41708e–006
81	0.00130627	0.0344541	0.000118209	1.00103e–006

method, the GLS method, and the multiscale enrichment method for the transient Navier–Stokes equations on uniform meshes. These five methods have the common discrete formulation: Find $(u_h(\cdot,t),p_h(\cdot,t)) \in (X_h,M_h)$, $t \in (0,T]$, such that

$$(u_{ht}, v_h) + a(u_h, v_h) - d(v_h, p_h) + b(u_h, u_h, v_h) = (f, v_h), \quad (6.1)$$
$$d(u_h, q_h) + \Lambda(p_h, q_h) = 0, \quad (6.2)$$

for all $(v_h,q_h) \in (X_h, M_h)$, where $\Lambda(p_h, q_h) = 0$ for the standard Galerkin method, $\Lambda(p_h, q_h) = \epsilon(p_h, q_h)/\nu$ for the penalty method, $\Lambda(p_h, q_h) = \delta\sum_K h_K^2(\nabla p_h - f, \nabla q_h)_K$ for the GLS method [7,24],

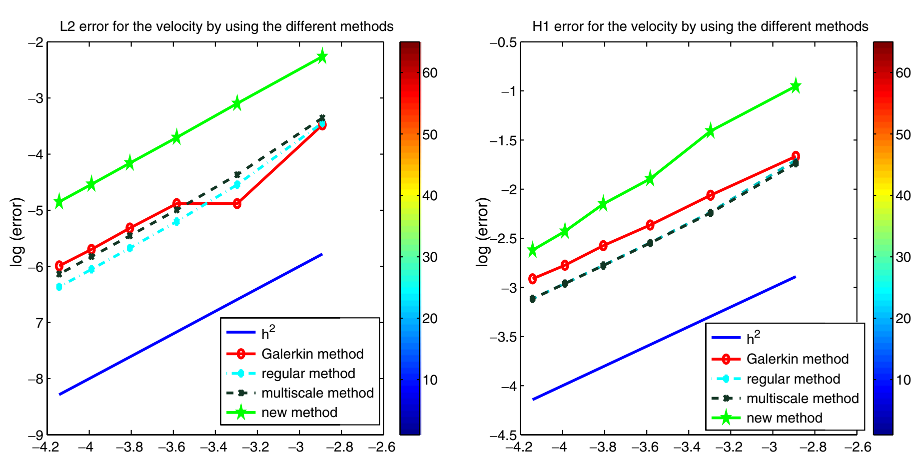
$$\Lambda(p_h, q_h) = \delta_1 \sum_K h_K^2(\nabla p_h - f, \nabla q_h)_K + \sum_{K,K'} \delta_2 h_e \left\langle \left[\nu\frac{\partial u}{\partial n}\right], \left[\nu\frac{\partial v}{\partial n}\right] \right\rangle_e,$$

for the multiscale enrichment method [2] and $\Lambda(p_h,q_h) = G(p_h,q_h)$ for the new method, where $h_e = |e|$ and $[v]$ denotes the jump of v across $e = K \cap K'$.

There is no satisfactory way to obtain the optimal parameters for the stabilized parameter with any given mesh. In practice, these parameters are still being determined by trial and error. The penalty method involves a stabilization parameter $1.0e-4 > \epsilon > 0$ [4,21], which must be sufficiently small. There is a slight deterioration in the finite element approximation by the regular stabilized method with the selected stabilized parameter in [16] for

Table 6
The results for the transient Navier–Stokes equations (new method with $\nu = 0.01$ on uniform mesh)

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	$\|\text{div}(u-u^h)\|_{0,\infty,K}$
18	0.104027	0.386244	0.00937275	0.000525787
27	0.0450949	0.244439	0.00391038	0.000162308
36	0.0246115	0.150347	0.00246092	6.65249e–005
45	0.0156104	0.116457	0.00156202	3.40361e–005
54	0.0107037	0.0881183	0.00115775	1.94697e–005
63	0.00781958	0.0727358	0.000866687	1.22037e–005
72	0.00595256	0.0604797	0.000690971	8.12688e–006
81	0.00468698	0.0520517	0.000560169	5.68476e–006



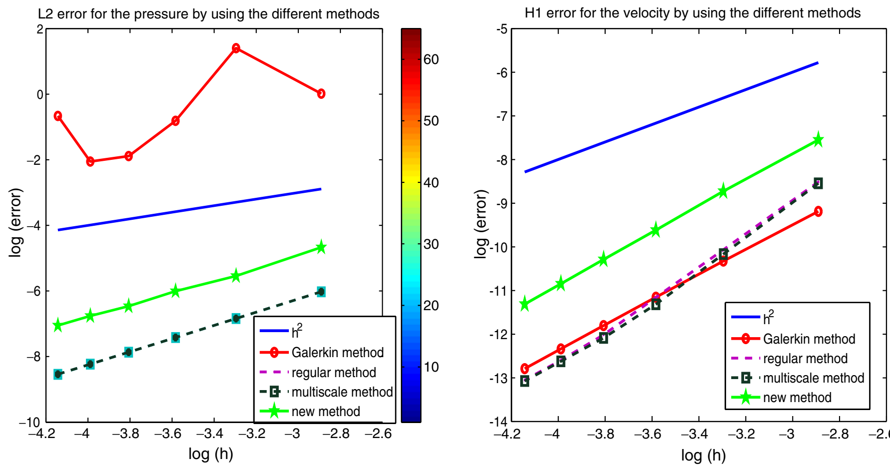


Fig. 2. Comparison of rate analysis for the transient Navier–Stokes equations by using four different methods ($v = 1.0e - 2$ on uniform mesh).

the stationary Navier–Stokes equations. Here, we choose $\epsilon = 1.0e - 6$, $\delta = 1/(80v) = 0.0125/v$, $\delta_1 = 1/(8v) = 0.125/v$, and $\delta_2 = 1/(12v)$ (see [2]). Furthermore, we test the effect of different stabilized parameters on the convergence rates, which are shown in Tables 1 and 2, with $h = 1/27$ and $v = 0.01$. We can clearly see that these two methods completely agree with the above choice of stabilized parameters in theory for the implicit scheme used.

Except for the penalty method, the relative convergence rates for the velocity and pressure approximations over different grids for the four methods are given in Tables 3–6 and Fig. 2. From our numerical experiments, there is not much difference among the four methods for the velocity approximation. However, as Table 3 shows, the difference is huge for the pressure approximation. The suboptimal error estimates for the standard Galerkin method look really bad, which is not surprising since this method does not satisfy the *inf-sup* condition for the $P_1 - P_1$ element used. The error estimates for the new method, the GLS method, and the multiscale enrichment method are more stable than those for the standard Galerkin method, which seems superconvergent. The results of the new stabilized method for the transient Navier–Stokes equations confirm the convergence theory developed in the previous sections. From Theorem 5.4, the theoretical analysis predicts a convergence rate of order $O(h)$ for velocity in the energy norm, $O(h^2)$ for velocity in the L^2-norm, and $O(h)$ for pressure in the L^2-norm. The numerical experiments in Table 6 show slightly better convergence rates. In fact, the convergence rates for pressure are close to an order of $O(h^{1.8})$. Most importantly, there is no negative effect with respect to the incompressible property by using the new stabilized method (see Tables 3–6 and Fig. 2). However, compared with the GLS method and multiscale enrichment method, the new stabilized method does not require the derivations or integration on the boundary of the set K. It only involves two local Gauss integrations.

We also design an unstructured mesh as shown in the second figure of Fig. 1. We take the solution of the transient Navier–Stokes equations with the Taylor-Hood $P_2 - P_1$ element over finer grid as the "exact" solution. Note that the same right-hand side with the previous numerical experiment on uniform meshes is used. It is no doubt that the stable Taylor-Hood element performs better since it employs the P_2 element for the velocity. Also, the Taylor-Hood element indicates its superconvergence performance. Then, the absolute error is obtained by comparing the "exact" solution and the finite element solutions with different methods. Finally, "error constants" can be obtained by a series of simple calculations (see Table 7). From Table 7, we can observe that the better "error constants" are obtained by the new stabilized finite element method.

Finally, the driven cavity is considered for the new stabilized finite element method over the unstructured mesh of Fig. 1. It is a box full of liquid with its lid moving horizontally at speed one. We compare the new stabilized finite element method with the GLS method, the multiscale method, and the standard Galerkin method with the stable Taylor-Hood element $P_2 - P_1$ and the mini-element $P_1 - P_1$. In particular, we plot the vertical component of the velocity along the horizontal line passing through the geometrical center of the cavity. The results for both velocity and pressure are given in Fig. 3. The numerical results of the new stabilized finite element method indicate the same performance as those of the standard finite element method with the Taylor-Hood element.

Table 7
Maximum value of the "error constants" ($v = 1.0e - 3$)

Type	$\max\left\{\frac{\lVert\bar{u}-u_h^m\rVert_0}{h^2}\right\}$	$\max\left\{\frac{\lVert\nabla(\bar{u}-u_h^m)\rVert_0}{h}\right\}$	$\max\left\{\frac{\lVert\bar{p}-p_h^m\rVert_0}{h}\right\}$
GLS method	0.752114	6.12452	6.09415
Enriched multiscale method	0.893946	6.22024	6.09433
$P_1b - P_1$	1.47453	16.8133	6.09413
New method	0.750659	5.9606	6.09587

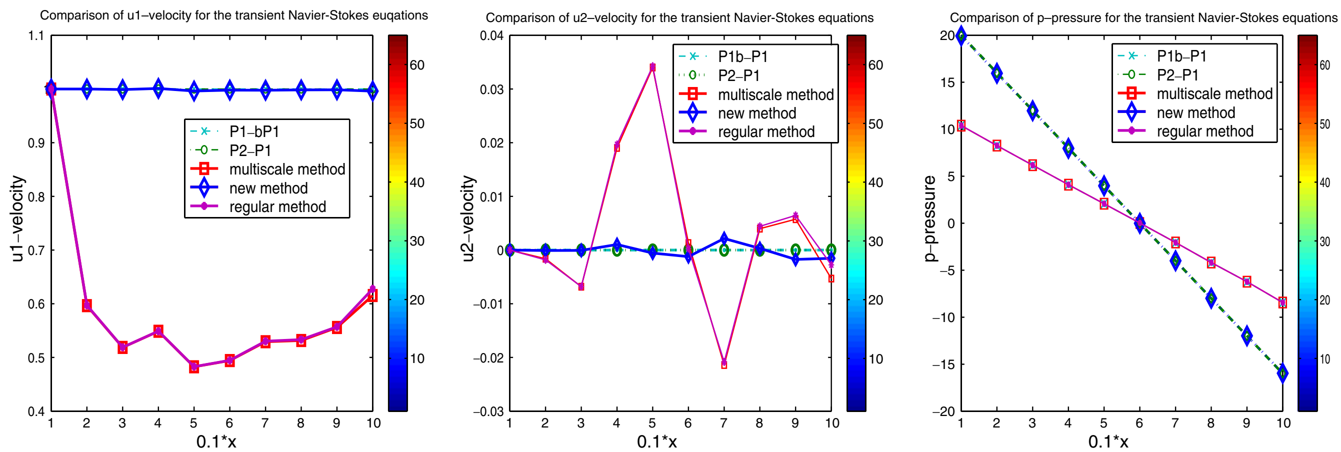


Fig. 3. Comparison of the velocity and pressure for the drive cavity problem by using five different method ($\nu = 1.0e - 3$ on unstructured mesh).

7. Conclusions

In this paper we have provided a theoretical analysis for a stabilized finite element method based on two local Gauss integrations. The analysis has extended the work in [20] for the stationary Stokes equations to the transient Navier–Stokes equations. The discretization uses a pair of spaces of finite elements $P_1 - P_1$ over triangles or $Q_1 - Q_1$ over quadrilateral elements. This new method is computationally efficient. It does not require a selection of mesh-dependent stabilization parameters or a calculation of higher-order derivatives. Its another valuable feature is that the action of stabilization operators can be performed locally at the element level with minimal additional cost. The numerical tests performed are in a good agreement with the theoretical results established. Most importantly, better performance of the new stabilized method than other stabilized methods on structured and unstructured mesh is observed.

Acknowledgement

We would like to express our sincere gratitude to Professors Pavel B. Bochev and Junping Wang for their valuable comments on an earlier version of this manuscript.

References

[2] R. Araya, G.R. Barrenechea, F. Valentin, Stabilized finite element methods based on multiscale enrichment for the Stokes problem, Numer. Math. 44 (2006) 322–348.
[3] C. Baiocchi, F. Brezzi, Stabilization of unstable numerical methods, in: Proc. Problemi Attuali Dell. Analisi e Della Fisica Matematica, Taormina, 1992, pp. 59–64.
[4] P. Bochev, M. Gunzburger, An absolutely stable pressure-Poisson stabilized finite element method for the Stokes equations, SIAM J. Numer. Anal. 42 (2004) 1189–1207.
[5] B. Brefort, J.M. Ghidaglia, R. Temam, Attractor for the penalty Navier–Stokes equations, SIAM J. Math. Anal. 19 (1988) 1–21.
[6] F. Brezzi, J. Douglas Jr., Stabilized mixed methods for the Stokes problem, Numer. Math. 53 (1988) 225–235.
[7] F. Brezzi, J.Pitkäranta, On the stabilisation of finite element approximations of the Stokes problem, in: W. Hackbusch (Ed.), Efficient Solutions of Elliptic systems, 1984, pp.11–19.
[8] G.C. Buscagliaa, F.G. Basombrio, R. Codina, Fourier analysis of an equal-order incompressible flow solver stabilized by pressure gradient projection, Int. J. Numer. Meth. Fluids 34 (2000) 65–92.
[9] Z. Chen, Finite Element Methods and their Applications, Spring-Verlag, Heidelberg, 2005.
[10] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[11] J. Douglas Jr., J. Wang, An absolutely stabilized finite element method for the Stokes problem, Math. Comp. 52 (1989) 495–508.
[12] R. Falk, An analysis of the penalty method and extrapolation for the stationary Stokes equations, in: R. Vichnevetsky (Ed.), Adv. Comput. Methods Partial Diff. Equat., AICA, 1975, pp. 66–69.
[13] K. Gerdes, D. SchöKtza, Hp-finite element simulations for Stokes flow- stable and stabilized, Finite Elements Anal. Des. 33 (1999) 143–165.
[14] V. Girault, P.A. Raviart, Finite Element Method for Navier–Stokes Equations: Theory and Algorithms, Springer-Verlag, Berlin and Heidelberg, 1987.
[16] Y. He, J. Li, A stabilized finite element method based on local polynomial pressure projection for the stationary Navier–Stokes equations, Appl. Numer. Math., in press.
[17] J.G. Heywood, R. Rannacher, Finite element approximation of the nonstationary Navier–Stokes problem I: Regularity of solutions and second-order error estimates for spatial discretization, SIAM J. Numer. Anal. 19 (1982) 275–311.
[18] A.T. Hill, E. Süli, Approximation of the global attractor for the incompressible Navier–Stokes problem, IMA J. Numer. Anal. 20 (2000) 633–667.
[19] T. Hughes, L. Franca, M. Balestra, A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuska–Brezzi condition: A stable Petrov–Galerkin formulation of the Stokes problem accommodating equal-order interpolations, Comput. Methods Appl. Mech. Engrg. 59 (1986) 85–99.
[20] J. Li, Y. He, A stabilized finite element method based on two local Gauss integrations for the Stokes equations, J. Comp. Appl. Math., in press.
[21] J. Li, L. Mei, Y. He, A pressure-Poisson stabilized finite element method for the non-stationary Stokes equations to circumvent the *inf-sup* condition, Appl. Math. Comp. 1 (2006) 24–35.
[22] J. Shen, On error estimates of the penalty method for unsteady Navier–Stokes equations, SIAM J. Numer. Anal. 32 (1995) 386–403.

[23] D.J. Silvester, Optimal low-order finite element methods for incompressible flow, Comp. Methods Appl. Mech. Engrg. 111 (1994) 357–368.
[24] D.J. Silvester, Stabilized mixed finite element methods, numerical analysis Report No. 262, Department of Mathematics, University of Manchester Institute of Science and Technology, Manchester, UK, 1995.
[25] D.J. Silvester, N. Kechkar, Stabilized bilinear-constant velocity-pressure finite elements for the conjugate gradient solution of the Stokes problem, Comp. Methods Appl. Mech. Engrg. 79 (1990) 71–86.
[26] B. Smith, P. Bjorstad, W. Gropp, Domain Decomposition, Parallel Multilevel Methods for Elliptic Partial Differential Equations, Cambridge University Press, Cambridge, 1996.
[27] R. Temam, Navier–Stokes Equations, third ed., Theory Numer. Anal., North-Holland, Amsterdam, 1983.